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Multidimensional modal analysis of nonlinear sloshing in a rectangular tank with finite water depth

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The discrete infinite-dimensional modal system describing nonlinear sloshing of an incompressible fluid with irrotational flow partially occupying a tank performing an arbitrary three-dimensional motion is derived in general form. The tank has vertical walls near the free surface and overturning waves are excluded. The derivation is based on the Bateman–Luke variational principle. The free surface motion and velocity potential are expanded in generalized Fourier series. The derived infinite-dimensional modal system couples generalized time-dependent coordinates of free surface elevation and the velocity potential. The procedure is not restricted by any order of smallness. The general multidimensional structure of the equations is approximated to analyse sloshing in a rectangular tank with finite water depth. The amplitude–frequency response is consistent with the fifth-order steady-state solutions by Waterhouse (1994). The theory is validated by new experimental results. It is shown that transients and associated nonlinear beating are important. An initial variation of excitation periods is more important than initial conditions. The theory is invalid when either the water depth is small or water impacts heavily on the tank ceiling. Alternative expressions for hydrodynamic loads are presented. The procedure facilitates simulations of a coupled vehicle–fluid system.

1. Introduction

A main objective is to describe violent fluid motions (sloshing) in a partly filled tank forced to oscillate in a frequency domain close to its lowest natural frequency. The ratio between maximum free surface amplitude and characteristic tank motion amplitude is then high and significant nonlinearities occur. This has practical interest for sloshing in ship tanks. By considering sea states that a ship has to operate in, it is realistic that wave-induced ship motions can cause resonant fluid oscillations. This can lead to large local structural loads in the tank and has an important effect on the global ship motions. It is desirable to develop numerical methods that accurately describe the fluid loading and coupling between ship motions and sloshing. A necessary requirement is that long time simulations can be performed and the proper statistical distribution of response variables obtained for various sea states.

Several studies on different numerical approaches to sloshing have been reported by Su Tsung-Chow (1992), Buechmann (1996), Tanizawa (1996), Chen *et al.* (1997), Pawell (1997) and the Loads Committee of the 13th ISSC (Moan & Berge 1997). A general drawback is the limited ability to perform long time simulations, especially for coupled 'liquid–structure' interactions. It may also be difficult to find water impact loads and local structural response. One reason is that water impact studies would often require a very fine discretization in time and space. Hydroelasticity may also have to be considered. We have instead focused on developing a semi-analytical method based on modal modelling. The present method assumes a smooth tank. This implies that potential theory can be used. The method also requires vertical tank walls near the mean free surface in its equilibrium position. Overturning waves cannot be described. It will be shown that a high degree of analysis can be performed. The consequences are both a time efficient and robust method. Water impact is not studied in detail, but the method can be combined with a local slamming analysis (see Faltinsen & Rognebakke 1999) and applied to coupled 'fluid–tank' simulations. An example is given illustrating the damping effect of forceful water impact on fluid motion.

Modal modelling of nonlinear sloshing implies that the equation of the free surface $\Sigma(t): z = f(x, y, t)$ is expanded in generalized Fourier series by a set of natural modes. The free surface elevation and the unknown velocity potential φ are expressed as

$$\left.\begin{aligned} z = f(x, y, t) &= \sum \beta_i(t)(\text{surfacemode})_i(x, y), \\ \varphi(x, y, z, t) &= \sum R_i(t)(\text{domainmode})_i(x, y, z). \end{aligned}\right\} \tag{1.1}$$

The (x, y, z) coordinate system is fixed relative to the tank; x, y are coordinates in the plane of the unperturbed water surface and t is the time variable. Generally speaking, the surface and volume modes are arbitrary known functions. However, they are typically chosen by the relation

$$(\text{surfacemode})_i(x, y) = (\text{domainmode})_i(x, y, z)|_{\Sigma_0}, \tag{1.2}$$

where Σ_0 is the unperturbed free surface. Since $f(x, y, t)$ is single-valued, (1.1) does not describe overturning waves. Moreover, $(\text{surfacemode})_i(x, y)$ must have a non-varying domain of definition. This means the tank must have vertical walls near the free surface in its equilibrium position.

The generalized coordinates β_i and R_i are found by a coupled system of nonlinear ordinary differential equations (modal system). The derivation of the modal system from the original free boundary problem was first proposed by Narimanov (1957) based on a perturbation technique. It has been further developed by Dodge, Kana & Abramson (1965), Narimanov, Dokuchaev & Lukovsky (1977) and Lukovsky (1990). These and other authors used a perturbation technique combined with variational (Hamilton–Ostrogradsky) projective method and derived small-dimensional models (1–3 degrees of freedom in a vertical circular cylinder) in the generalized coordinates $\beta_i(t)$ or their averaged values (for resonantly excited waves). (See for instance Lukovsky 1976; Miles (1976, 1984*a*, *b*).) Using an averaging technique means that β_i is written as

$$\beta_i = \sum_{j=0}^{\infty} (\langle\beta_i\rangle_j^1(\tau)\sin(j\sigma t) + \langle\beta_i\rangle_j^2(\tau)\cos(j\sigma t)),$$

where σ is the excitation frequency and τ is slowly varying relative to t. The averaged equations of a $\langle\beta_i\rangle^j(\tau)$ have the form of a Duffing equation for a rectangular tank (see Shemer 1990 and Tsai, Yue & Yip 1990) or a system of four first-order ordinary differential equations for a vertical circular cylindrical tank (see Miles (1984*a*, *b*)). Funakoshi & Inoue (1991) used Miles' model in their detailed simulations. The averaging technique and small-dimensional modal modelling complement each another in the analysis of the steady-state free surface response due to periodic tank excitations. But these methods are questionable in modelling coupled fluid–structure interaction with complicated non-periodic tank motions when transient effects matter. These complex motions are simulated in engineering applications either numerically or by phenomenological (usually pendulum) models (see Chapter 5 of Narimanov *et al.* 1977 or Pilipchuk & Ibrahim 1997). An alternative is to use Narimanov's original technique with the modal representation in the form (1.1) and more general asymptotic assumptions of β_i and R_n in order to reach reasonable dimensions of the modal systems. The successful use of this approach is reported by Limarchenko & Yasinsky (1997) and Lukovsky & Timokha (1995) for simplified models of spacecraft. A similar method was used by Ikeda & Nakagawa (1997) for analysis of damping of vessel motions due to sloshing. This suggests that multidimensional modal analysis can simulate complicated nonlinear wave phenomena coupled with structural vibrations.

The general form of a discrete infinite-dimensional modal system is derived in the first part of this paper by the Bateman–Luke (pressure-integral Lagrangian) variational principle. This idea was proposed independently by Miles (1976) and Lukovsky (1976). They studied forced small-amplitude translatory motions of a vertical circular cylindrical tank. The surfacemodes and domainmodes were obtained by linear theory and related by (1.2). Our derivation of a discrete infinite-dimensional modal system is not restricted to a particular type of body motion. The surface and domain modes are not associated with natural modes and no asymptotic assumptions are introduced in the first stage of the derivation. The infinite-dimensional modal system can be reduced to a finite-dimensional form by assuming small-amplitude forced oscillations and associate order of magnitudes of the different modes. This is done in the second part of the paper to analyse nonlinear sloshing in a two-dimensional rectangular smooth tank with finite water depth. Both forced translatory and rotational body motions are considered. The lowest natural mode is assumed to dominate and the three lowest modes interact nonlinearly with each other. Several modes having higher order are considered by linear theory. The asymptotic theory constructed is a special multidimensional analogue of the model by Ikeda & Nakagawa (1997) and the direct generalization of the third-order hydrodynamic theory by Faltinsen (1974).

Experiments on nonlinear sloshing caused by primary mode resonant excitation have been conducted. The asymptotic modal theory constructed explains the basic observed phenomena including modulated ('beating') waves with a high accuracy of amplitude and 'beating period' characteristics. The beating is a consequence of transients that do not die out on a very long time scale. The reason is the very small damping of the fluid motion inside a smooth tank with no internal structures obstructing the flow and no heavy water impact on the tank ceiling. A consequence is that steady-state response of the fluid motion can have a limited capability to describe sloshing quantitatively. However the steady-state response is valuable to understand important features of the flow like stability and how the response is influenced by water depth, excitation frequency and amplitude. Since it represents a special case of our theory, steady-state solutions are used in the verification process. Examples of steady-state amplitude–frequency response for surge- and pitch-excited nonlinear waves are presented. The results are consistent with the third- and fifth-order steady-state solutions by Faltinsen (1974) and Waterhouse (1994) respectively.

The use of a discrete modal system allows us to calculate various kinematic and dynamic characteristics occurring due to interaction between the fluid and the tank. We present examples of hydrodynamic force and moment on the tank. The structure of the equations describing the fluid motion as a function of the rigid body motions makes it possible to set up an equation system for the coupled tank and fluid motion. An example could be analysis of a ship tank due to wave-induced ship motions. Since the wave conditions that cause violent sloshing may not be extreme, we can use linear time domain theory to describe external hydrodynamic loads acting on the ship. By setting up the equations of motions for the global rigid ship motions together with the equations describing sloshing, complex fluid–structure interaction can be analysed. But the theory does not describe the effect of impact on the tank ceiling. This can easily occur in practical applications and is an area of further research. The asymptotic theory is not applicable to shallow water. This is due to secondary parametric resonance and means that the primary mode is not dominating. The ratio between water depth and breadth of the tank is 0.173 in the example where the finite water depth theory does not work. This is not really shallow water in a hydrodynamic context (see the nonlinear theory by Verhagen & Wijngaarden 1965). What we need is a theory that can combine the present finite water depth theory with a nonlinear shallow water theory.

2. Free boundary value problem

A mobile rigid tank partly filled by an inviscid incompressible fluid is considered. The flow is irrotational. The fluid volume bounded by the free surface $\Sigma(t)$ and the wetted tank surface $S(t)$ is denoted $Q(t)$. Let $O'x'y'z'$ be an absolute coordinate system and $Oxyz$ be a moving coordinate system fixed with respect to the rigid tank. The origin of $Oxyz$ is in the unperturbed free surface and moves with the velocity $\boldsymbol{v}_0$ relative to $O'x'y'z'$. The tank has an angular velocity $\boldsymbol{\omega}$ relative to $O'x'y'z'$. The gravity field has the potential

$$U(x, y, z, t) = -\boldsymbol{g} \cdot \boldsymbol{r}', \qquad \boldsymbol{r}' = \boldsymbol{r}'_0 + \boldsymbol{r}, \tag{2.1}$$

where $\boldsymbol{r}'$ is the radius-vector of a point of the body–fluid system with respect to O', $\boldsymbol{r}'_0$ is the radius-vector of the point O with respect to O', $\boldsymbol{r}$ is the radius-vector with respect to O and $\boldsymbol{g}$ is the gravity acceleration vector.

Since the flow is irrotational, the fluid velocity can be represented as $\boldsymbol{v}_a = \nabla\Phi$, where $\boldsymbol{v}_a$ is the fluid velocity vector at the point (x, y, z) in the moving coordinate system and $\Phi(x, y, z, t)$ is the velocity potential. The velocity potential and the free surface $\Sigma(t)$ can be found from the following nonlinear free boundary problem:

$$\left.\begin{aligned}
&\Delta\Phi = 0 \quad \text{in } Q(t), \qquad \frac{\partial\Phi}{\partial\nu} = \boldsymbol{v}_0 \cdot \boldsymbol{\nu} + \boldsymbol{\omega} \cdot [\boldsymbol{r} \times \boldsymbol{\nu}] \quad \text{on } S(t),\\
&\frac{\partial\Phi}{\partial\nu} = \boldsymbol{v}_0 \cdot \boldsymbol{\nu} + \boldsymbol{\omega} \cdot [\boldsymbol{r} \times \boldsymbol{\nu}] + \frac{\xi_t}{|\nabla\xi|} \quad \text{on } \Sigma(t),\\
&\frac{\partial\Phi}{\partial t} + \tfrac{1}{2}(\nabla\Phi)^2 - \nabla\Phi \cdot (\boldsymbol{v}_0 + \boldsymbol{\omega} \times \boldsymbol{r}) + U = 0 \quad \text{on } \Sigma(t), \qquad \int_{Q(t)} \mathrm{d}Q = \text{const.}
\end{aligned}\right\} \tag{2.2}$$

Here $\boldsymbol{\nu}$ is the outer normal to the boundary of $Q(t)$ and $\xi(x, y, z, t) = 0$ is the equation of the free surface $\Sigma(t)$. The last integral condition in (2.2) implies fluid volume conservation and is also the well-known solvability condition for the Neumann boundary value problem.

The free boundary problem (2.2) must be completed by initial or periodicity conditions to get a unique solution. The first type of condition introduces the initial position of the free surface $\Sigma(t_0)$ and the initial distribution of normal derivatives of Φ, i.e.

$$\xi(t_0, x, y, z) = \xi_0(x, y, z), \qquad \left.\frac{\partial \Phi}{\partial \nu}\right|_{\Sigma(t_0)} = \phi(x, y, z). \tag{2.3}$$

Here $\xi_0(x, y, z)$ and $\phi(x, y, z)$ are given functions. If the flow starts from rest with sufficiently small tank oscillations, linear theory can be used to formulate the initial conditions. One way of doing this is in terms of impulse conservation. This means

$$\Phi = 0 \quad \text{on } \Sigma_0 \quad \text{and} \quad \text{zero free surface elevation for } t = t_0. \tag{2.4}$$

The last free surface boundary condition (dynamic boundary condition) of (2.2) is obtained by using Lagrange–Cauchy integral for the pressure in the moving coordinate system. It states that the pressure on the free surface is equal to a constant p_0. The hydrodynamic pressure p in $Q(t)$ can be obtained by

$$\frac{\partial \Phi}{\partial t} + \tfrac{1}{2}(\nabla \Phi)^2 - \nabla \Phi \cdot (\boldsymbol{v}_0 + \boldsymbol{\omega} \times \boldsymbol{r}) + U + \frac{p - p_0}{\rho} = 0 \quad \text{in } Q(t). \tag{2.5}$$

Here $\partial \Phi / \partial t$ is calculated in the moving coordinate system, i.e. for a point rigidly connected with the system $Oxyz$.

There is a set of mechanical characteristics (expressed by integrals of Φ and its derivatives), which describes the interaction between the vessel and fluid. They are:

(a) the radius-vectors of the mass centre with respect to the points O' and O ($\boldsymbol{r}'_{1C}$ and $\boldsymbol{r}_{1C}$) $\boldsymbol{r}'_{1C} = \boldsymbol{r}'_0 + \boldsymbol{r}_{1C}$, where

$$\rho \int_{Q(t)} U \,\mathrm{d}Q = -\rho \int_{Q(t)} \boldsymbol{g} \cdot \boldsymbol{r}' \,\mathrm{d}Q = -m_1 \boldsymbol{g} \cdot \boldsymbol{r}'_{1C};$$

(b) the resulting hydrodynamic forces $\boldsymbol{F}(t)$, and moments $\boldsymbol{N}(t)$ on the tank

$$\boldsymbol{F}(t) = \int_{S(t)} (p - p_0)\boldsymbol{\nu} \,\mathrm{d}S, \qquad \boldsymbol{N}(t) = \int_{S(t)} \boldsymbol{r} \times ((p - p_0)\boldsymbol{\nu}) \,\mathrm{d}S. \tag{2.6}$$

3. Derivation of the general modal system by the variational method

Let us consider the boundary value problem (2.2). The unknowns are $\Phi = \Phi(x, y, z, t)$ and $\xi(x, y, z, t)$. We will use a Bateman–Luke variational principle and introduce the pressure in the Lagrangian of the Hamilton principle. The idea of the pressure integral as the Lagrangian in hydrodynamic problems was first proposed by Hargneaves (1908). The canonical formulation of this principle is given by Bateman (1944) and Luke (1967) (for gravity surface waves in infinite basins). We use the formulation given by Lukovsky (1990).

Pressure-integral Lagrangian variational principle. *The boundary value problem given by (2.2) can be described by examining the necessary conditions for the extrema of the functional*

$$W = \int_{t_1}^{t_2} L \,\mathrm{d}t, \tag{3.1}$$

where the Lagrangian L is the pressure integral

$$L = \int_{Q(t)} (p - p_0)\,\mathrm{d}Q = -\rho \int_{Q(t)} \left[\frac{\partial \Phi}{\partial t} + \tfrac{1}{2}(\nabla\Phi)^2 - \nabla\Phi \cdot (\boldsymbol{v}_0 + \boldsymbol{\omega} \times \boldsymbol{r}) + U \right] \mathrm{d}Q; \quad (3.2)$$

and the test functions satisfy

$$\delta\Phi(x, y, z, t_1) = 0, \quad \delta\Phi(x, y, z, t_2) = 0; \quad \delta\xi(x, y, z, t_1) = 0, \quad \delta\xi(x, y, z, t_2) = 0. \quad (3.3)$$

We consider a domain Q having vertical walls in a neighbourhood of the free surface in the equilibrium position. The normal velocity component on the free surface $z = f(x, y, t)$ is given in the body-fixed system by $-\xi_t/|\nabla\xi| = f_t/\sqrt{1 + f_x^2 + f_y^2}$. The velocity potential is expressed as

$$\Phi(x, y, z, t) = \boldsymbol{v}_0 \cdot \boldsymbol{r} + \boldsymbol{\omega} \cdot \boldsymbol{\Omega} + \varphi. \quad (3.4)$$

The vector-function $\boldsymbol{\Omega}(x, y, z) = (\Omega_1, \Omega_2, \Omega_3)$ (Stokes–Zhukovsky potentials) is the solution of the following Neumann boundary value problem:

$$\left.\begin{aligned} &\Delta\boldsymbol{\Omega} = 0 \quad \text{in } Q(t), \\ &\left.\frac{\partial \Omega_1}{\partial \nu}\right|_{S(t)+\Sigma(t)} = y\nu_3 - z\nu_2, \quad \left.\frac{\partial \Omega_2}{\partial \nu}\right|_{S(t)+\Sigma(t)} = z\nu_1 - x\nu_3, \\ &\left.\frac{\partial \Omega_3}{\partial \nu}\right|_{S(t)+\Sigma(t)} = x\nu_2 - y\nu_1, \end{aligned}\right\} \quad (3.5)$$

where ν_1, ν_2, ν_3 are the projections of the outer normal $\boldsymbol{\nu}$ onto the $Oxyz$-axes. The function φ is a solution of the Neumann boundary value problem

$$\Delta\varphi = 0 \quad \text{in } Q(t), \quad \left.\frac{\partial \varphi}{\partial \nu}\right|_{S(t)} = 0, \quad \left.\frac{\partial \varphi}{\partial \nu}\right|_{\Sigma(t)} = \frac{f_t}{\sqrt{1 + f_x^2 + f_y^2}}.$$

The Neumann boundary value problems for $\boldsymbol{\Omega}$ and φ have unique solutions since

$$\int_{S(t)+\Sigma(t)} \frac{\partial \Omega_i}{\partial \nu}\,\mathrm{d}S = 0, \quad \int_{\Sigma(t)} \frac{\partial \varphi}{\partial \nu}\,\mathrm{d}S = \int_{\Sigma(t)} \frac{f_t}{\sqrt{1 + (\nabla f)^2}}\,\mathrm{d}S = 0$$

are always fulfilled (see Lukovsky & Timokha 1995). These solutions depend parametrically on time. By using (3.4) and the boundary value problems for $\boldsymbol{\Omega}$ and φ it follows that $\boldsymbol{\Phi}$ satisfies the Laplace equation and the Neumann boundary conditions of (2.2). The dynamic condition (pressure balance) on $\Sigma(t)$ gives the final equation connecting $f, \boldsymbol{\Omega}$ and φ.

Let the function $f(x, y, t)$ be expressed as

$$f(x, y, t) = \sum_{i=1}^{\infty} \beta_i(t) f_i(x, y), \quad (3.6)$$

where $f_i(x, y)$ is a complete (to within a constant) orthogonal system of functions satisfying the condition of volume conservation $\int_{\Sigma_0} f_i(x, y)\,\mathrm{d}x\,\mathrm{d}y = 0$. Further,

$$\varphi(x, y, z, t) = \sum_{n=1}^{\infty} R_n(t)\,\varphi_n(x, y, z), \quad (3.7)$$

where the complete system of functions $\varphi_n(x, y, z)$ satisfies the Laplace equation in the whole tank domain Q and zero Neumann boundary condition on the wetted body surface. Normally, only the wetted body surface below the mean free surface is considered. Since the system $\{\varphi_n(x, y, z)\}$ is complete on any single-connected surface in the tank domain, it is also a complete system on Σ_0. The Stokes–Zhukovsky potentials Ω_i are assumed to be known functions of β_i. Hence, we must only find the unknown functions $\beta_i(t)$ and $R_n(t)$.

Such a family of harmonic functions $\varphi_n(x, y, z)$ can be chosen as a set of solutions of the following boundary spectral problems with spectral parameter λ_n:

$$\Delta\varphi_n = 0 \quad \text{in } Q_0, \quad \frac{\partial\varphi_n}{\partial\nu} = 0 \quad \text{on } S, \quad \frac{\partial\varphi_n}{\partial\nu} = \lambda_n\varphi_n \quad \text{on } \Sigma_0, \quad \int_{\Sigma_0} \varphi_n \, \mathrm{d}S = 0. \tag{3.8}$$

This is the same as the linear eigenvalue problem for sloshing. The solutions can be found analytically only for a limited class of tank shapes. Examples are a vertical circular cylinder or a rectangular three-dimensional tank. However, a numerical method can be used to find φ_n for a general tank shape. This was demonstrated by Solaas & Faltinsen (1997), where Moiseev's theory was applied to two-dimensional sloshing. A different approach is to use a patching procedure and consider for instance a tank consisting of a cylindrical part near the free surface. Then the solution in the cylindrical part can be expressed as

$$\varphi_n(x, y, z) = \sum_k (b_{nk} \exp(-\lambda_k z) + a_{nk} \exp(\lambda_k z))\phi_k(x, y) \tag{3.9}$$

with unknown coefficients b_{nk} and a_{nk}. Here λ_k and ϕ_k are the solutions of the following spectral problem:

$$\Delta_2\phi_k(x, y) + \lambda_k^2\phi_k = 0 \quad \text{in } \Sigma_0, \quad \frac{\partial\phi_k}{\partial n} = 0 \quad \text{on } \partial\Sigma_0, \quad \int_{\Sigma_0} \phi_k \, \mathrm{d}S = 0, \tag{3.10}$$

where $\partial\Sigma_0$ is the intersection line between Σ_0 and S. The solution in the non-cylindrical part can be found by a numerical method. When the auxiliary problem (3.10) is formulated in circular (ring-shaped) or rectangular cross-sections Σ_0, the solutions ϕ_k of (3.10) are expressed by Bessel functions and/or sinusoidal functions. Otherwise, a numerical procedure for (3.10) is required.

By substituting (3.4) into (3.2) the Lagrangian L takes the following form:

$$\begin{aligned} L = -\rho \int_{Q(t)} \Big[& \dot{\boldsymbol{v}}_0 \cdot \boldsymbol{r} + \frac{\partial}{\partial t}(\boldsymbol{\omega} \cdot \boldsymbol{\Omega}) + \tfrac{1}{2}\nabla(\boldsymbol{\omega} \cdot \boldsymbol{\Omega}) \cdot \nabla(\boldsymbol{\omega} \cdot \boldsymbol{\Omega}) - \boldsymbol{\omega} \cdot (\boldsymbol{r} \times \nabla(\boldsymbol{\omega} \cdot \boldsymbol{\Omega})) \\ & - \tfrac{1}{2}v_0^2 - \boldsymbol{\omega} \cdot (\boldsymbol{r} \times \boldsymbol{v}_0) - \boldsymbol{\omega} \cdot (\boldsymbol{r} \times \nabla\varphi) + \nabla(\boldsymbol{\omega} \cdot \boldsymbol{\Omega}) \cdot \nabla\varphi \Big] \mathrm{d}Q + L_r, \end{aligned} \tag{3.11}$$

where

$$L_r = -\rho \int_{Q(t)} \left[\frac{\partial\varphi}{\partial t} + \tfrac{1}{2}(\nabla\varphi)^2 + U \right] \mathrm{d}Q. \tag{3.12}$$

The two last integrand terms in square brackets of (3.11) cancel each other from Green's formula, i.e.

$$\int_{Q(t)} (\nabla(\boldsymbol{\omega} \cdot \boldsymbol{\Omega}) \cdot \nabla\varphi - (\boldsymbol{\omega} \times \boldsymbol{r}) \cdot \nabla\varphi) \, \mathrm{d}Q = \int_{S(t)+\Sigma(t)} \left(\frac{\partial(\boldsymbol{\omega} \cdot \boldsymbol{\Omega})}{\partial\nu} - (\boldsymbol{\omega} \times \boldsymbol{r}) \cdot \boldsymbol{\nu} \right) \varphi \, \mathrm{d}S = 0.$$

We also introduce the quadratic symmetric inertia tensor $\boldsymbol{J}^1$ with components J_{ij}^1

defined by the equality

$$-\rho\int_{Q(t)}\left(\tfrac{1}{2}\nabla(\boldsymbol{\omega}\cdot\boldsymbol{\Omega})\cdot\nabla(\boldsymbol{\omega}\cdot\boldsymbol{\Omega})-\boldsymbol{\omega}\cdot(\boldsymbol{r}\times\nabla(\boldsymbol{\omega}\cdot\boldsymbol{\Omega}))\right)\mathrm{d}Q$$
$$=-\tfrac{1}{2}\omega_1^2J_{11}^1-\tfrac{1}{2}\omega_2^2J_{22}^1-\tfrac{1}{2}\omega_3^2J_{33}^1-\omega_1\omega_2J_{12}^1-\omega_1\omega_3J_{13}^1-\omega_2\omega_3J_{23}^1.$$

These components J_{ij}^1 can be calculated by Green's formula, i.e.

$$J_{11}^1=\rho\int_{Q(t)}\left(y\frac{\partial\Omega_1}{\partial z}-z\frac{\partial\Omega_1}{\partial y}\right)\mathrm{d}Q=\rho\int_{S(t)+\Sigma(t)}\Omega_1\frac{\partial\Omega_1}{\partial\nu}\,\mathrm{d}S, \tag{3.13a}$$

$$J_{22}^1=\rho\int_{Q(t)}\left(z\frac{\partial\Omega_2}{\partial x}-x\frac{\partial\Omega_2}{\partial z}\right)\mathrm{d}Q=\rho\int_{S(t)+\Sigma(t)}\Omega_2\frac{\partial\Omega_2}{\partial\nu}\,\mathrm{d}S, \tag{3.13b}$$

$$J_{33}^1=\rho\int_{Q(t)}\left(x\frac{\partial\Omega_3}{\partial y}-y\frac{\partial\Omega_3}{\partial x}\right)\mathrm{d}Q=\rho\int_{S(t)+\Sigma(t)}\Omega_3\frac{\partial\Omega_3}{\partial\nu}\,\mathrm{d}S, \tag{3.13c}$$

$$J_{12}^1=J_{21}^1=\rho\int_{Q(t)}\left(z\frac{\partial\Omega_1}{\partial x}-x\frac{\partial\Omega_1}{\partial z}\right)\mathrm{d}Q=\rho\int_{Q(t)}\left(y\frac{\partial\Omega_2}{\partial z}-z\frac{\partial\Omega_2}{\partial y}\right)\mathrm{d}Q$$
$$=\rho\int_{S(t)+\Sigma(t)}\Omega_1\frac{\partial\Omega_2}{\partial\nu}\,\mathrm{d}S=\rho\int_{S(t)+\Sigma(t)}\Omega_2\frac{\partial\Omega_1}{\partial\nu}\,\mathrm{d}S, \tag{3.13d}$$

$$J_{13}^1=J_{31}^1=\rho\int_{Q(t)}\left(x\frac{\partial\Omega_1}{\partial y}-y\frac{\partial\Omega_1}{\partial x}\right)\mathrm{d}Q=\rho\int_{Q(t)}\left(y\frac{\partial\Omega_3}{\partial z}-z\frac{\partial\Omega_3}{\partial y}\right)\mathrm{d}Q$$
$$=\rho\int_{S(t)+\Sigma(t)}\Omega_1\frac{\partial\Omega_3}{\partial\nu}\,\mathrm{d}S=\rho\int_{S(t)+\Sigma(t)}\Omega_3\frac{\partial\Omega_1}{\partial\nu}\,\mathrm{d}S, \tag{3.13e}$$

$$J_{23}^1=J_{32}^1=\rho\int_{Q(t)}\left(x\frac{\partial\Omega_2}{\partial y}-y\frac{\partial\Omega_2}{\partial x}\right)\mathrm{d}Q=\rho\int_{Q(t)}\left(z\frac{\partial\Omega_3}{\partial x}-x\frac{\partial\Omega_3}{\partial z}\right)\mathrm{d}Q$$
$$=\rho\int_{S(t)+\Sigma(t)}\Omega_2\frac{\partial\Omega_3}{\partial\nu}\,\mathrm{d}S=\rho\int_{S(t)+\Sigma(t)}\Omega_3\frac{\partial\Omega_2}{\partial\nu}\,\mathrm{d}S. \tag{3.13f}$$

The Lagrangian L (3.11) can be rewritten as

$$\begin{aligned}L=-\,&[\dot{v}_{01}l_1+\dot{v}_{02}l_2+\dot{v}_{03}l_3+\dot{\omega}_1l_{1\omega}+\dot{\omega}_2l_{2\omega}+\dot{\omega}_3l_{3\omega}+\omega_1l_{1\omega t}+\omega_2l_{2\omega t}\\&+\omega_3l_{3\omega t}-\tfrac{1}{2}(\omega_1^2J_{11}^1+\omega_2^2J_{22}^1+\omega_3^2J_{33}^1)-\omega_1\omega_2J_{12}^1-\omega_1\omega_3J_{13}^1\\&-\omega_2\omega_3J_{23}^1-\tfrac{1}{2}m_1(v_{01}^2+v_{02}^2+v_{03}^2)+(\omega_2v_{03}-\omega_3v_{02})l_1\\&+(\omega_3v_{01}-\omega_1v_{03})l_2+(\omega_1v_{02}-\omega_2v_{01})l_3]+L_r,\end{aligned} \tag{3.14}$$

where

$$\left.\begin{aligned}&m_1=\rho\int_{Q(t)}\mathrm{d}Q,\quad l_{k\omega}=\rho\int_{Q(t)}\Omega_k\,\mathrm{d}Q,\quad l_{k\omega t}=\rho\int_{Q(t)}\frac{\partial\Omega_k}{\partial t}\,\mathrm{d}Q,\\&l_1=\rho\int_{Q(t)}x\,\mathrm{d}Q,\quad l_2=\rho\int_{Q(t)}y\,\mathrm{d}Q,\quad l_3=\rho\int_{Q(t)}z\,\mathrm{d}Q.\end{aligned}\right\} \tag{3.15}$$

The vectors $\boldsymbol{l}=\{l_k\},\boldsymbol{l}_\omega=\{l_{k\omega}\},\boldsymbol{l}_{\omega t}=\{l_{k\omega t}\}$ depend only on $\beta_i(t)$ and $\dot{\beta}_i(t)$.

It follows from (3.7) that

$$L_r = -\rho \int_{Q(t)} \left[\sum_{n=1} \dot{R}_n \varphi_n + \tfrac{1}{2} \sum_n \sum_k R_n R_k (\nabla\varphi_n, \nabla\varphi_k) + U \right] \mathrm{d}Q$$
$$= -\left[\sum_n A_n \dot{R}_n + \tfrac{1}{2} \sum_n \sum_k A_{nk} R_n R_k - g_1 l_1 - g_2 l_2 - g_3 l_3 - m_1 \boldsymbol{g} \cdot \boldsymbol{r}_0' \right], \quad (3.16)$$

where

$$A_n = \rho \int_{Q(t)} \varphi_n \,\mathrm{d}Q, \quad A_{nk} = A_{kn} = \rho \int_{Q(t)} (\nabla\varphi_n, \nabla\varphi_k)\,\mathrm{d}Q = \rho \int_{\Sigma(t)+S(t)} \varphi_n \frac{\partial \varphi_k}{\partial \nu}\,\mathrm{d}S \quad (3.17)$$

are functions of $\beta_i(t)$.

The Lagrangian L is originally a function of two independent variables $f(x, y, z, t)$ and $\Phi(x, y, z, t)$. The independent variables become the time-varying functions $\beta_i(t), i \geqslant 1$ and $R_n(t), n \geqslant 1$ after substituting (3.4), (3.6) and (3.7) into the Lagrangian. The variations of the functional (3.1) by $\beta_i(t)$ and $R_n(t)$ for given $\boldsymbol{v}_0(t)$ and $\boldsymbol{\omega}(t)$ are

$$\delta W = \int_{t_1}^{t_2} \left[\sum_n A_n \delta \dot{R}_n + \sum_n \sum_k A_{nk} R_k \delta R_n + \sum_i \left(\sum_n \dot{R}_n \frac{\partial A_n}{\partial \beta_i} \right.\right.$$
$$+\omega_1 \frac{\partial l_{1\omega t}}{\partial \beta_i} + \omega_2 \frac{\partial l_{2\omega t}}{\partial \beta_i} + \omega_3 \frac{\partial l_{3\omega t}}{\partial \beta_i} + \tfrac{1}{2} \sum_n \sum_k R_n R_k \frac{\partial A_{nk}}{\partial \beta_i}$$
$$+\dot{\omega}_1 \frac{\partial l_{1\omega}}{\partial \beta_i} + \dot{\omega}_2 \frac{\partial l_{2\omega}}{\partial \beta_i} + \dot{\omega}_3 \frac{\partial l_{3\omega}}{\partial \beta_i} + (\dot{v}_{01} - g_1 + \omega_2 v_{03} - \omega_3 v_{02}) \frac{\partial l_1}{\partial \beta_i}$$
$$+(\dot{v}_{02} - g_2 + \omega_3 v_{01} - \omega_1 v_{03}) \frac{\partial l_2}{\partial \beta_i} + (\dot{v}_{03} - g_3 + \omega_1 v_{02} - \omega_2 v_{01}) \frac{\partial l_3}{\partial \beta_i}$$
$$\left. -\tfrac{1}{2}\omega_1^2 \frac{\partial J_{11}^1}{\partial \beta_i} - \tfrac{1}{2}\omega_2^2 \frac{\partial J_{22}^1}{\partial \beta_i} - \tfrac{1}{2}\omega_3^2 \frac{\partial J_{33}^1}{\partial \beta_i} - \omega_1\omega_2 \frac{\partial J_{12}^1}{\partial \beta_i} - \omega_1\omega_3 \frac{\partial J_{13}^1}{\partial \beta_i} - \omega_2\omega_3 \frac{\partial J_{23}^1}{\partial \beta_i} \right) \delta\beta_i$$
$$\left. + \left(\omega_1 \frac{\partial l_{1\omega t}}{\partial \dot{\beta}_i} + \omega_2 \frac{\partial l_{2\omega t}}{\partial \dot{\beta}_i} + \omega_3 \frac{\partial l_{3\omega t}}{\partial \dot{\beta}_i} \right) \delta\dot{\beta}_i \right] \mathrm{d}t = 0. \quad (3.18)$$

The following infinite system of nonlinear differential equations (*modal system*) coupling modal functions $R_n(t)$ and $\beta_i(t)$ is obtained by integrating by parts in (3.18) and using (3.3) for test functions:

$$\frac{\mathrm{d}}{\mathrm{d}t} A_n - \sum_k R_k A_{nk} = 0, \quad n = 1, 2, \ldots, \quad (3.19)$$

$$\sum_n \dot{R}_n \frac{\partial A_n}{\partial \beta_i} + \tfrac{1}{2} \sum_n \sum_k \frac{\partial A_{nk}}{\partial \beta_i} R_n R_k + \dot{\omega}_1 \frac{\partial l_{1\omega}}{\partial \beta_i} + \dot{\omega}_2 \frac{\partial l_{2\omega}}{\partial \beta_i} + \dot{\omega}_3 \frac{\partial l_{3\omega}}{\partial \beta_i} + \omega_1 \frac{\partial l_{1\omega t}}{\partial \beta_i} + \omega_2 \frac{\partial l_{2\omega t}}{\partial \beta_i}$$
$$+\omega_3 \frac{\partial l_{3\omega t}}{\partial \beta_i} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\omega_1 \frac{\partial l_{1\omega t}}{\partial \dot{\beta}_i} + \omega_2 \frac{\partial l_{2\omega t}}{\partial \dot{\beta}_i} + \omega_3 \frac{\partial l_{3\omega t}}{\partial \dot{\beta}_i} \right) + (\dot{v}_{01} - g_1 + \omega_2 v_{03} - \omega_3 v_{02}) \frac{\partial l_1}{\partial \beta_i}$$
$$+(\dot{v}_{02} - g_2 + \omega_3 v_{01} - \omega_1 v_{03}) \frac{\partial l_2}{\partial \beta_i} + (\dot{v}_{03} - g_3 + \omega_1 v_{02} - \omega_2 v_{01}) \frac{\partial l_3}{\partial \beta_i} - \tfrac{1}{2}\omega_1^2 \frac{\partial J_{11}^1}{\partial \beta_i}$$
$$-\tfrac{1}{2}\omega_2^2 \frac{\partial J_{22}^1}{\partial \beta_i} - \tfrac{1}{2}\omega_3^2 \frac{\partial J_{33}^1}{\partial \beta_i} - \omega_1\omega_2 \frac{\partial J_{12}^1}{\partial \beta_i} - \omega_1\omega_3 \frac{\partial J_{13}^1}{\partial \beta_i} - \omega_2\omega_3 \frac{\partial J_{23}^1}{\partial \beta_i} = 0. \quad (3.20)$$

The system of ordinary differential equations (3.19) can be considered as a linear system of algebraic equations $\sum A_{nk}(\beta_i)R_k = (\mathrm{d}/\mathrm{d}t)A_n(\beta_i)$. By using a numerical or asymptotic technique we can then find a solution of R_n as a function of β_i. After substituting R_n into (3.20) we get a system of second-order nonlinear differential equations with respect to β_i. The values $\partial l_k/\partial\beta_i$ are given by

$$\frac{\partial l_3}{\partial\beta_i} = \rho\int_{\Sigma_0} f_i^2\,\mathrm{d}S\ \beta_i = \lambda_{i3}\beta_i,\quad \frac{\partial l_2}{\partial\beta_i} = \rho\int_{\Sigma_0} y f_i\,\mathrm{d}S = \lambda_{i2},\quad \frac{\partial l_1}{\partial\beta_i} = \rho\int_{\Sigma_0} x f_i\,\mathrm{d}S = \lambda_{i1}. \tag{3.21}$$

The constructed infinite-dimensional system of equations (3.19) (3.20) is applicable to any type of rigid body motion. It is necessary that $f(x,y,t)$ given by (3.6) is single-valued. This means that plunging breakers cannot be described. There are no other restrictions on the type of surface wave that can be studied.

4. Modal system for two-dimensional fluid flows

We assume two-dimensional fluid motion in the (x,z)-plane. Then

$$\boldsymbol{v}_0 = (v_{0x}, 0, v_{0z}),\quad \boldsymbol{\omega} = (0, \omega(t), 0),\quad \boldsymbol{r} = (x, 0, z),\quad \boldsymbol{\Omega}(x,0,z) = (0, \Omega(x,z,t), 0) \tag{4.1}$$

and Ω is the solution of the following boundary value problem:

$$\Delta\Omega = 0 \quad \text{in } Q(t),\qquad \left.\frac{\partial\Omega}{\partial\nu}\right|_{S(t)+\Sigma(t)} = z\nu_1 - x\nu_3. \tag{4.2}$$

The velocity potential $\Phi(x,0,z,t)$ takes the form

$$\Phi(x,0,z,t) = v_{0x}x + v_{0z}z + \omega(t)\,\Omega(x,z,t) + \sum_{n=1}^{\infty} R_n(t)\,\varphi_n(x,z), \tag{4.3}$$

where $\varphi_n(x,z)$ is a complete system of harmonic functions satisfying the zero Neumann condition on the bottom and vertical walls and the Laplace equation in Q.

The general infinite-dimensional modal system of ordinary differential equations (3.19), (3.20) has in two dimensions the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t}A_n - \sum_k R_k A_{nk} = 0,\quad n = 1, 2, \ldots, \tag{4.4}$$

$$\sum_n \dot{R}_n \frac{\partial A_n}{\partial\beta_i} + \tfrac{1}{2}\sum_n\sum_k \frac{\partial A_{nk}}{\partial\beta_i} R_n R_k + \dot{\omega}\frac{\partial l_{2\omega}}{\partial\beta_i} + \omega\frac{\partial l_{2\omega t}}{\partial\beta_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\omega\frac{\partial l_{2\omega t}}{\partial\dot{\beta}_i}\right)$$
$$+(\dot{v}_{01} - g_1 + \omega v_{03})\frac{\partial l_1}{\partial\beta_i} + (\dot{v}_{03} - g_3 - \omega v_{01})\frac{\partial l_3}{\partial\beta_i} - \tfrac{1}{2}\omega^2\frac{\partial J_{22}^1}{\partial\beta_i} = 0. \tag{4.5}$$

5. Asymptotic modal system for a rectangular tank performing arbitrary small-amplitude motions

We consider a mobile rectangular rigid tank filled partly by an inviscid incompressible fluid. The mean water depth is h and l is the tank breadth. The flow is irrotational and two-dimensional (see figure 1). The origin of the coordinate system is in the mean free surface at the centreplane of the tank. The equation $z = f(x,t)$ determines the perturbed free surface $\Sigma(t)$. The fluid domain is

$$Q(t) = \{(x,z) : -h < z < f(x,t); -l/2 < x < l/2\}. \tag{5.1}$$

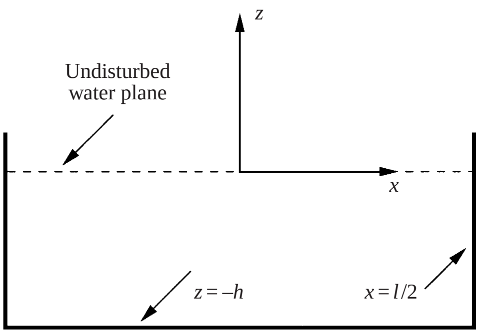

FIGURE 1. Coordinate system.

Since $f(x,t)$ is expressed by (3.6), the complete (to within a constant) orthogonal system of functions $\{f_i(x)\}$ should satisfy the volume conservation condition

$$\int_{-l/2}^{l/2} f_i(x)\,\mathrm{d}x = 0. \tag{5.2}$$

The modal system (4.4), (4.5) can be approximated to surface waves with one primary dominating mode corresponding to the first natural mode. This implies that the body motions are horizontal and/or rotational and quasi-periodic with average frequency close to the first resonance frequency. It is also necessary that the water depth is not shallow and the fluid does not hit the tank ceiling (see, also, physical arguments presented in Faltinsen 1974 and the book by Mikishev 1978). The rigid body motions are assumed small relative to the tank breadth and water depth.

The derivation of the finite-dimensional asymptotic analogue of the system (4.4) and (4.5) requires an asymptotic relation between dominating mode amplitude and excitation amplitude. It is assumed, as in the theory by Faltinsen (1974), that

$$O(\beta_1^3) = O(H) = O(\psi_0) = \epsilon. \tag{5.3}$$

Here H is translatory (surge) motion magnitude and ψ_0 is angular (pitch) magnitude. Further $\beta_2 = O(\epsilon^{2/3}), \beta_3 = O(\epsilon)$. Higher-order terms than ϵ will be neglected in the nonlinear equations. The modes $f_i(x)$ in (3.6) as well as $\varphi_i(x,z)$ in (3.7) can be chosen as the solutions of the spectral problem

$$\left.\begin{aligned} &\Delta\varphi_i = 0 \quad (-l/2 < x < l/2, -h < z < 0),\\ &\left.\frac{\partial\varphi_i}{\partial x}\right|_{x=-l/2, x=l/2} = 0, \quad \left.\frac{\partial\varphi_i}{\partial z}\right|_{z=-h} = 0, \quad \frac{\partial\varphi_i}{\partial z} = \lambda_i\varphi_i \quad (z=0). \end{aligned}\right\} \tag{5.4}$$

This means

$$\left.\begin{aligned} &\lambda_i = \frac{\pi \mathrm{i}}{l}\tanh\left(\frac{\mathrm{i}\pi}{l}h\right), \quad f_i(x) = \cos\left(\frac{\pi\mathrm{i}}{l}(x+l/2)\right),\\ &\varphi_i(x,z) = f_i(x)\frac{\cosh\left((\pi\mathrm{i}/l)(z+h)\right)}{\cosh\left((\pi\mathrm{i}/l)h\right)}. \end{aligned}\right\} \tag{5.5}$$

Equations (3.6) and (3.7) take the following form:

$$f(x,t) = \sum_{i=1}^{\infty}\beta_i(t)f_i(x), \quad \varphi(x,z,t) = \sum_{i=1}^{\infty} R_i(t) f_i(x)\frac{\cosh\left((\mathrm{i}\pi/l)(z+h)\right)}{\cosh\left((\mathrm{i}\pi/l)h\right)}. \tag{5.6}$$

By accounting for the asymptotic relation (5.3) and keeping only terms up to ϵ in the modal system (4.4) and (4.5) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}A_n - \sum_k R_k A_{nk} = 0, \quad n = 1, 2, \ldots, \tag{5.7}$$

$$\sum_n \dot{R}_n \frac{\partial A_n}{\partial \beta_i} + \tfrac{1}{2}\sum_n \sum_k \frac{\partial A_{nk}}{\partial \beta_i} R_n R_k + \dot{\omega}\frac{\partial l_{2\omega}}{\partial \beta_i} + \omega \frac{\partial l_{2\omega t}}{\partial \beta_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\omega \frac{\partial l_{2\omega t}}{\partial \dot{\beta}_i}\right) + (\dot{v}_{01} - g_1)\lambda_{i1} + (\dot{v}_{03} - g_3)\beta_i \lambda_{i3} = 0. \tag{5.8}$$

Asymptotic expansions of integrals $A_i, A_{nk}, l_2\omega, l_{2\omega t}$ have to be used in (5.7) and (5.8). Here $A_i, A_{nk}, l_{2\omega}, l_{2\omega t}$ are defined by (3.17) and (3.15) as integrals over the instantaneous fluid volume position. The integrals are divided into integrals over the mean position of fluid volume Q_0 and over the remaining part Q_δ. Q_δ is described by β_i. Further, the integrand of the integrals over Q_δ can be expanded in Taylor series by β_i. Keeping terms up to ϵ gives

$$\left.\begin{aligned} A_1 &= \frac{\rho l}{2}(\beta_1 + E_1(\beta_1\beta_2 + \beta_2\beta_3) + E_0(\beta_1^3 + 2\beta_1\beta_2^2 + \beta_1^2\beta_3)), \\ A_2 &= \frac{\rho l}{2}(\beta_2 + E_2(\beta_1^2 + 2\beta_1\beta_3) + 8E_0\beta_1^2\beta_2), \\ A_3 &= \frac{\rho l}{2}(\beta_3 + 3E_3\beta_1\beta_2 + 3E_0\beta_1^3); \end{aligned}\right\} \tag{5.9}$$

$$\left.\begin{aligned} &A_{11} = \rho l(E_1 + 8E_1E_0\beta_1^2 - (2E_0 - E_1^2)\beta_2), \quad A_{22} = \rho l(2E_2), \\ &A_{12} = A_{21} = \rho l((4E_0 + 2E_1E_2)\beta_1 + (-4E_0 + 2E_1^2)\beta_3), \\ &A_{33} = \rho l(3E_3), \quad A_{13} = A_{31} = 3l\rho(2E_0 + E_1E_3)(\beta_2 + 2E_4\beta_1^2), \\ &A_{23} = A_{32} = 3l\rho(4E_0 + 2E_2E_3)\beta_1, \end{aligned}\right\} \tag{5.10}$$

where

$$E_0 = \frac{1}{8}\left(\frac{\pi}{l}\right)^2, \quad E_i = \frac{\pi}{2l}\tanh\left(\frac{\pi i}{l}h\right), \quad i \geqslant 1. \tag{5.11}$$

Further, we express R_n as

$$R_n = \sum_i \gamma_i \dot{\beta}_i + \sum_{ij} \gamma_{ij}\dot{\beta}_j\beta_i + \sum_{ijk}\gamma_{ijk}\dot{\beta}_i\beta_j\beta_k + \cdots$$

and substitute it in (5.7). Explicit values of $\gamma_i, \gamma_{ij}, \gamma_{ijk}$ are found by collecting similar terms. The result is

$$\left.\begin{aligned} &R_1 = \frac{\dot{\beta}_1}{2E_1} + \frac{E_0}{E_1^2}\dot{\beta}_1\beta_2 - \frac{E_0}{E_1E_2}\dot{\beta}_2\beta_1 + \frac{E_0}{E_1}\left(-\tfrac{1}{2} + \frac{4E_0}{E_1E_2}\right)\beta_1^2\dot{\beta}_1, \\ &R_2 = \frac{1}{4E_2}\left(\dot{\beta}_2 - \frac{4E_0}{E_1}\beta_1\dot{\beta}_1\right), \quad R_3 = \frac{\dot{\beta}_3}{6E_3} - \frac{E_0}{E_1E_3}\dot{\beta}_1\beta_2 - \frac{E_0}{E_2E_3}\dot{\beta}_2\beta_1 \\ &+\dot{\beta}_1\beta_1^2\left(\frac{3E_2}{2E_3} - \frac{2E_0E_4}{E_1E_3} - E_4 + \frac{4E_0^2}{E_1E_2E_3} + \frac{2E_0E_2}{E_1E_3}\right), \quad R_i = \frac{\dot{\beta}_i}{2iE_i}, \quad i \geqslant 4; \end{aligned}\right\} \tag{5.12}$$

and

$$\left.\begin{aligned}\dot{R}_1 &= \frac{\ddot{\beta}_1}{2E_1} + \frac{E_0}{E_1^2}\ddot{\beta}_1\beta_2 - \frac{E_0}{E_1E_2}\ddot{\beta}_2\beta_1 + \dot{\beta}_1\dot{\beta}_2\left(\frac{E_0}{E_1^2} - \frac{E_0}{E_1E_2}\right)\\ &\quad + \frac{E_0}{E_1}\left(-\frac{1}{2} + \frac{4E_0}{E_1E_2}\right)\beta_1^2\ddot{\beta}_1 + 2\frac{E_0}{E_1}\left(-\tfrac{1}{2} + \frac{4E_0}{E_1E_2}\right)\dot{\beta}_1^2\beta_1,\\ \dot{R}_2 &= \frac{1}{4E_2}\left(\ddot{\beta}_2 - \frac{4E_0}{E_1}(\beta_1\ddot{\beta}_1 + \dot{\beta}_1^2)\right),\\ \dot{R}_3 &= \frac{\ddot{\beta}_3}{6E_3} - \frac{E_0}{E_1E_3}\ddot{\beta}_1\beta_2 - \frac{E_0}{E_2E_3}\ddot{\beta}_2\beta_1 - \left(\frac{E_0}{E_1E_3} + \frac{E_0}{E_2E_3}\right)\dot{\beta}_1\dot{\beta}_2 + (\ddot{\beta}_1\beta_1^2 + 2\dot{\beta}_1^2\beta_1)\\ &\quad \times\left(\frac{3E_2}{2E_3} - \frac{2E_0E_4}{E_1E_3} - E_4 + \frac{4E_0^2}{E_1E_2E_3} + \frac{2E_0E_2}{E_1E_3}\right), \qquad \dot{R}_i = \frac{\ddot{\beta}_1}{2iE_i}, \quad i \geqslant 4.\end{aligned}\right\} \tag{5.13}$$

By calculating λ_{ij} we get

$$\left.\begin{aligned}\lambda_{i1} &= \rho\int_{-l/2}^{l/2} x\cos\left(\frac{\mathrm{i}\pi}{l}(x+l/2)\right)\mathrm{d}x = \rho\left(\frac{l}{\mathrm{i}\pi}\right)^2((-1)^i - 1),\\ \lambda_{i3} &= \rho\int_{-l/2}^{l/2}\cos^2\left(\frac{\mathrm{i}\pi}{l}(x+l/2)\right)\mathrm{d}x = \frac{\rho l}{2}.\end{aligned}\right\} \tag{5.14}$$

$l_{2\omega}$ and $l_{2\omega t}$ (see (3.15)) depend on $\Omega(x,z,t)$ which is the solution of the boundary value problem (4.2). $\Omega(x,z,t)$ depends parametrically on $\beta_i(t)$ due to the free surface $\Sigma(t)$. Since $\partial l_{2\omega}/\partial\beta_i$ and $\partial l_{2\omega t}/\partial\beta_i$ are multiplied by terms of $O(\epsilon)$ in (5.8), it is sufficient to include only linear terms in β_i in the integrals $l_{2\omega}$ and $l_{2\omega t}$. The problem (4.2) in a rectangular tank takes the following form:

$$\left.\begin{aligned}&\Delta\Omega = 0 \quad \text{in } Q(t), \quad \frac{\partial\Omega}{\partial z} = -x \quad (z = -h),\\ &\frac{\partial\Omega}{\partial x} = z\left(x = \frac{l}{2}, -\frac{l}{2}\right), \quad \frac{\partial\Omega}{\partial v} = -x\frac{1}{\sqrt{1+(f_x)^2}} - z\frac{f_x}{\sqrt{1+(f_x)^2}} \quad (z = f(x,t)).\end{aligned}\right\} \tag{5.15}$$

The solution can be found by a Zhukovsky-type substitution with additional terms for fluctuations of the free surface. This gives

$$\Omega = xz - 2\sum_{i=1}^{\infty} a_i f_i \frac{\sinh((\pi/l)\mathrm{i}(z+h/2))}{\cosh((\pi/2l)\mathrm{i}h)} + \sum_{i=1}^{\infty}\chi_i(t) f_i \frac{\cosh((\pi/l)\mathrm{i}(z+h))}{\cosh((\pi/l)\mathrm{i}h)}. \tag{5.16}$$

The coefficients a_i are found from the condition $\chi_i(t) \equiv 0$, $i \geqslant 1$, if and only if, $\beta_i \equiv 0$, $i \geqslant 1$. Substitution of (5.16) into (5.15) gives

$$\sum_{i=1}^{N} a_i f_i \frac{\mathrm{i}\pi}{l} = x \quad \text{or} \quad a_i = \frac{2l^2}{(\mathrm{i}\pi)^3}[(-1)^i - 1]. \tag{5.17}$$

The functions $\chi_i(t)$ follow from (5.15) after substitution of (5.16) and (5.17) and performing the Taylor series technique for the free surface $\Sigma(t)$ (with respect to β_i). The linear terms of $l_{2\omega}$ and $l_{2\omega t}$ do not depend on $\chi_i(t)$. To show this we substitute

(5.16) in the corresponding integrals

$$l_{2\omega} = -2\rho \sum_{i=1}^{\infty} a_i \tanh\left(\frac{\mathrm{i}\pi}{2l}h\right) \beta_i \int_{-l/2}^{l/2} f_i^2 \,\mathrm{d}x + \rho \sum_{i=1}^{\infty} \chi_i(t) \frac{l}{\mathrm{i}\pi} \tanh\left(\frac{\mathrm{i}\pi}{l}h\right) \int_{-l/2}^{l/2} f_i \,\mathrm{d}x, \tag{5.18}$$

$$l_{2\omega t} = \rho \sum_{j=1}^{\infty} \dot{\chi}(t) \frac{l}{\mathrm{i}\pi} \tanh\left(\frac{\mathrm{i}\pi}{l}h\right) \int_{-l/2}^{l/2} f_i \,\mathrm{d}x. \tag{5.19}$$

It follows from the volume conservation condition (5.2) that

$$l_{2\omega t} = 0, \quad l_{2\omega} = -2\rho \sum_{i=1}^{\infty} \beta_i \left(\frac{l}{\mathrm{i}\pi}\right)^3 [(-1)^i - 1] \tanh\left(\frac{\mathrm{i}\pi}{2l}h\right). \tag{5.20}$$

The derivatives with respect to β_i give

$$\frac{\partial l_{2\omega t}}{\partial \beta_i} = 0, \quad \frac{\partial l_{2\omega}}{\partial \beta_i} = -2\rho \left(\frac{l}{\mathrm{i}\pi}\right)^3 [(-1)^i - 1] \tanh\left(\frac{\mathrm{i}\pi}{2l}h\right), \quad i \geqslant 1. \tag{5.21}$$

Finally, by defining the angular position of the mobile coordinate system $Oxyz$ with respect to $O'x'y'z'$ as $\psi(t)$ we obtain correct to $O(\epsilon)$ that

$$g_3 = -g, \quad g_1 = g\psi(t). \tag{5.22}$$

The terms in (5.8) $\ddot{\psi}\partial l_{2\omega}/\partial\beta_i + (-g_3)\beta_i\lambda_{3i} + (-g_1)\lambda_{1i}$ caused by forced pitch excitation can be rewritten as

$$-\rho \left(\frac{l}{\mathrm{i}\pi}\right)^2 [(-1)^i - 1] \left(\frac{2l}{\mathrm{i}\pi} \tanh\left(\frac{\mathrm{i}\pi}{2l}h\right) \ddot{\psi}(t) + g\psi(t)\right) + g\beta_i. \tag{5.23}$$

When substituting above formula in (5.8), we get the following system of ordinary differential equations describing modal oscillations of a fluid in a rectangular tank performing arbitrary small-magnitude motions (keeping terms up to third order in the nonlinear equations):

$$(\ddot{\beta}_1 + \sigma_1^2\beta_1) + d_1(\ddot{\beta}_1\beta_2 + \dot{\beta}_1\dot{\beta}_2) + d_2(\ddot{\beta}_1\beta_1^2 + \dot{\beta}_1^2\beta_1) + d_3\ddot{\beta}_2\beta_1 + P_1(\dot{v}_{0x} - S_1\dot{\omega} - g\psi) + Q_1\dot{v}_{0z}\beta_1 = 0, \tag{5.24a}$$

$$(\ddot{\beta}_2 + \sigma_2^2\beta_2) + d_4\ddot{\beta}_1\beta_1 + d_5\dot{\beta}_1^2 + Q_2\dot{v}_{0z}\beta_2 = 0, \tag{5.24b}$$

$$(\ddot{\beta}_3 + \sigma_3^2\beta_3) + d_6\ddot{\beta}_1\beta_2 + d_7\ddot{\beta}_1\beta_1^2 + d_8\ddot{\beta}_2\beta_1 + d_9\dot{\beta}_1\dot{\beta}_2 + d_{10}\dot{\beta}_1^2\beta_1 + P_3(\dot{v}_{0x} - S_3\dot{\omega} - g\psi) + Q_3\dot{v}_{0z}\beta_3 = 0. \tag{5.24c}$$

The linear equations describing higher modes are

$$\ddot{\beta}_i + \sigma_i^2\beta_i + P_i(\dot{v}_{0x} - S_i\dot{\omega} - g\psi) + Q_i\dot{v}_{0z}\beta_i = 0, \quad i \geqslant 4. \tag{5.25}$$

Here v_{0x} and v_{0z} are projections of the translational velocity onto axes of Oxz, $\omega(t)$ is the value of the angular velocity of coordinate system $Oxyz$ with respect to $O'x'y'z'$.

The coefficients introduced are calculated by formulas

$$\sigma_i^2 = 2g\mathrm{i}E_i, \quad P_{2i-1} = -\frac{8E_{2i-1}l}{\pi^2(2\mathrm{i}-1)}, \quad P_{2i} = 0, \quad Q_i = 2\mathrm{i}E_i,$$
$$S_i = \frac{2l}{\pi\mathrm{i}} \tanh\left(\frac{\mathrm{i}\pi}{2l}h\right), \quad i \geqslant 1, \tag{5.26}$$

where σ_i is the natural frequency of mode i. Further,

$$\left.\begin{aligned} & d_1 = 2\frac{E_0}{E_1} + E_1, \quad d_2 = 2E_0\left(-1 + \frac{4E_0}{E_1E_2}\right), \quad d_3 = -2\frac{E_0}{E_2} + E_1, \\ & d_4 = -4\frac{E_0}{E_1} + 2E_2, \quad d_5 = E_2 - 2\frac{E_0E_2}{E_1^2} - \frac{4E_0}{E_1}, \quad d_6 = 3E_3 - \frac{6E_0}{E_1}, \\ & d_7 = 9E_0 - 12\frac{E_0E_4}{E_1} - 6E_3E_4 + 24\frac{E_0^2}{E_1E_2} + 3\frac{E_0E_3}{E_1}, \\ & d_8 = -6\frac{E_0}{E_2} + 3E_3, \quad d_9 = -6\frac{E_0}{E_1} - 6\frac{E_0}{E_2} - 6\frac{E_0E_3}{E_1E_2} + 3\frac{E_3E_1}{E_2}, \\ & d_{10} = 18E_0 - 2E_4\frac{12E_0 + 6E_1E_3}{E_1} + \frac{72E_0^2}{E_1E_2} + 12E_0\left(\frac{E_3}{E_1} - \frac{E_1}{E_2}\right). \end{aligned}\right\} \tag{5.27}$$

The first two nonlinear equations of (5.24) couple β_1 with β_2 and do not depend on β_3. The third mode component is excited by rigid body motions and the first and the second modes of sloshing. The second mode response becomes infinite if the excitation has frequency content at the natural frequency for the second mode; and similarly for the third and higher modes. The first mode will be finite if it is excited at the natural frequency of the first mode. This is caused by nonlinear effects and will become more evident in the next section on steady-state response.

6. Steady-state sloshing in a rectangular tank with a small-amplitude surge/pitch sinusoidal excitation

The theory of steady-state solutions of the nonlinear sloshing problem in a rectangular tank was created by Faltinsen (1974) based on Moiseev's (1958) method. The constructed asymptotic discrete theory (5.24) makes it possible to generalize the main relations of this theory. For surge-excited steady-state waves we express $\boldsymbol{v}_0$ as $(-H\sigma\sin(\sigma t), 0, 0)$, set $\omega = \psi \equiv 0$ and look for periodic solutions

$$\beta_i(t + 2\pi/\sigma) = \beta_i(t), \qquad \dot{\beta}_i(t + 2\pi/\sigma) = \dot{\beta}_i(t) \tag{6.1}$$

of the discrete model (5.24).

To construct asymptotically the periodic solutions and to derive analytically the amplitude–frequency response of nonlinear sloshing in a rectangular tank caused by forced excitation we express the first approximation of the primary mode in the form

$$\beta_1(t) = A\cos\sigma t + o(A). \tag{6.2}$$

The substitution of (6.2) into (5.24*b*) with periodicity condition (6.1) yields

$$\beta_1 = A\cos\sigma t + o(A), \qquad \beta_2 = A^2(l_0 + h_0\cos(2\sigma t)) + o(A^2), \tag{6.3}$$

where

$$l_0 = \frac{d_4 - d_5}{2\bar{\sigma}_2^2}, \qquad h_0 = \frac{d_5 + d_4}{2(\bar{\sigma}_2^2 - 4)}, \qquad \bar{\sigma}_i = \frac{\sigma_i}{\sigma}, \quad i = 1, 2. \tag{6.4}$$

The amplitude $A \sim \epsilon^{1/3}$ of the primary mode can be found by substituting (6.3) into the first equation of (5.24) and collecting Fourier terms of lowest order. The equation coupling primary mode amplitude, frequency, breadth and depth will be non-dimensionalized by dividing all length variables by l. This gives

$$\Pi_h(\bar{\sigma}_1, \bar{\sigma}_2, \bar{A}) = (\bar{\sigma}_1^2 - 1)\bar{A} + m_1(\bar{\sigma}_2, \bar{h})\bar{A}^3 - \bar{P}_1\bar{H} = 0, \tag{6.5}$$

$$m_1(\bar{\sigma}_2, \bar{h}) = \bar{d}_1(-\bar{l}_0(\bar{\sigma}_2) + \tfrac{1}{2}\bar{h}_0(\bar{\sigma}_2)) - \tfrac{1}{2}\bar{d}_2 - 2\bar{d}_3\bar{h}_0(\bar{\sigma}_2), \tag{6.6}$$

where the overbar denotes non-dimensionalized value. The coefficient m_1 in equation (6.5) depends on depth–breadth ratio and frequency of excitation ($\bar{\sigma}_i, i = 1, 2$). The last dependence has not been presented earlier for frequency–amplitude response equations. Usually, the corresponding coefficient depends only on h/l. This means that our asymptotic technique differs from Faltinsen–Moiseev's procedure. In order to compare both techniques we need to give the following remark.

Remark. For any asymptotic theory with one dominating mode the nonlinear equation describing the dependence of amplitude–breadth ratio $\bar{A}$ and frequency of excitation σ has the same general form

$$\Pi_h\left(\frac{\sigma_1}{\sigma}, \frac{\sigma_2}{\sigma}, \bar{A}\right) = 0,$$

where σ_1 is the natural frequency of the primary mode.

The function Π can be expanded in a Taylor series. The approach by Moiseev (1958) and Faltinsen (1974) gives the expansion near the point $(\bar{\sigma}_1, \sigma_2/\sigma_1, 0)$ (for fixed $\bar{h}$). Our approach has no asymptotic restriction on the value of frequency σ and, therefore, includes only power series in $\bar{A}$

$$\Pi_h\left(\frac{\sigma_1}{\sigma}, \frac{\sigma_2}{\sigma}, \bar{A}\right) = \Pi\left(\frac{\sigma_1}{\sigma}, \frac{\sigma_2}{\sigma}, 0\right) + \frac{\partial \Pi}{\partial \bar{A}}\left(\frac{\sigma_1}{\sigma}, \frac{\sigma_2}{\sigma}, 0\right)\bar{A} + \frac{1}{2}\frac{\partial^2 \Pi}{\partial \bar{A}^2}\left(\frac{\sigma_1}{\sigma}, \frac{\sigma_2}{\sigma}, 0\right)\bar{A}^2 + \frac{\partial^3 \Pi}{\partial \bar{A}^3}\left(\frac{\sigma_1}{\sigma}, \frac{\sigma_2}{\sigma}, 0\right)\tfrac{1}{6}\bar{A}^3 + o(\bar{A}^3).$$

Since the value σ_2/σ is used to calculate only m_1 we in fact make a more precise calculation of this coefficient.

Equation (6.5) gives infinite response for $\bar{\sigma}_1 = 1$ and $m_1 = 0$. It implies that the third-order theory is not valid if

$$m_1\left(\frac{\sigma_2}{\sigma_1}, \bar{h}\right) = 0.$$

The root of the last equation gives $\bar{h} = h/l = 0.3374\ldots$. This is called the critical depth and coincides with the known value (see Waterhouse 1994). The response changes from being a 'hard-spring' to a 'soft-spring' at the critical depth. The detailed asymptotic analysis of the response near critical depth was done by Waterhouse (1994) by fifth-order theory based on Faltinsen–Moiseev's technique. It was shown that the branches in the amplitude–frequency plane coincide with a third-order theory only for small amplitude. New turning points on the branches occur at a critical value of the amplitude/frequency.

In our case $m_1 = m_1(\sigma_2/\sigma, \bar{h})$ which means that m_1 is a function of σ and $\bar{h}$. If a fixed σ is close to the natural frequency σ_1, but $\sigma \neq \sigma_1$ the equation

$$m_1\left(\frac{\sigma_2}{\sigma}, \bar{h}\right) = 0 \qquad (6.7)$$

gives a different value of the critical depth. This means that the critical depth is a function of σ. If a pair $(\sigma, \bar{h})$ satisfies (6.7), then $\bar{A}$ tends to infinity. This effect is illustrated in figures 2 and 3.

Figure 2 shows the positive and negative solutions (branches P_+, P_-) of the secular algebraic equation (6.5) for different values of the water depth h and fixed amplitude of excitation H. The choice of H corresponds to the experimental values reported later in the paper. Branch O is the set of solutions of (6.5) for $H = 0$ (no vibration of

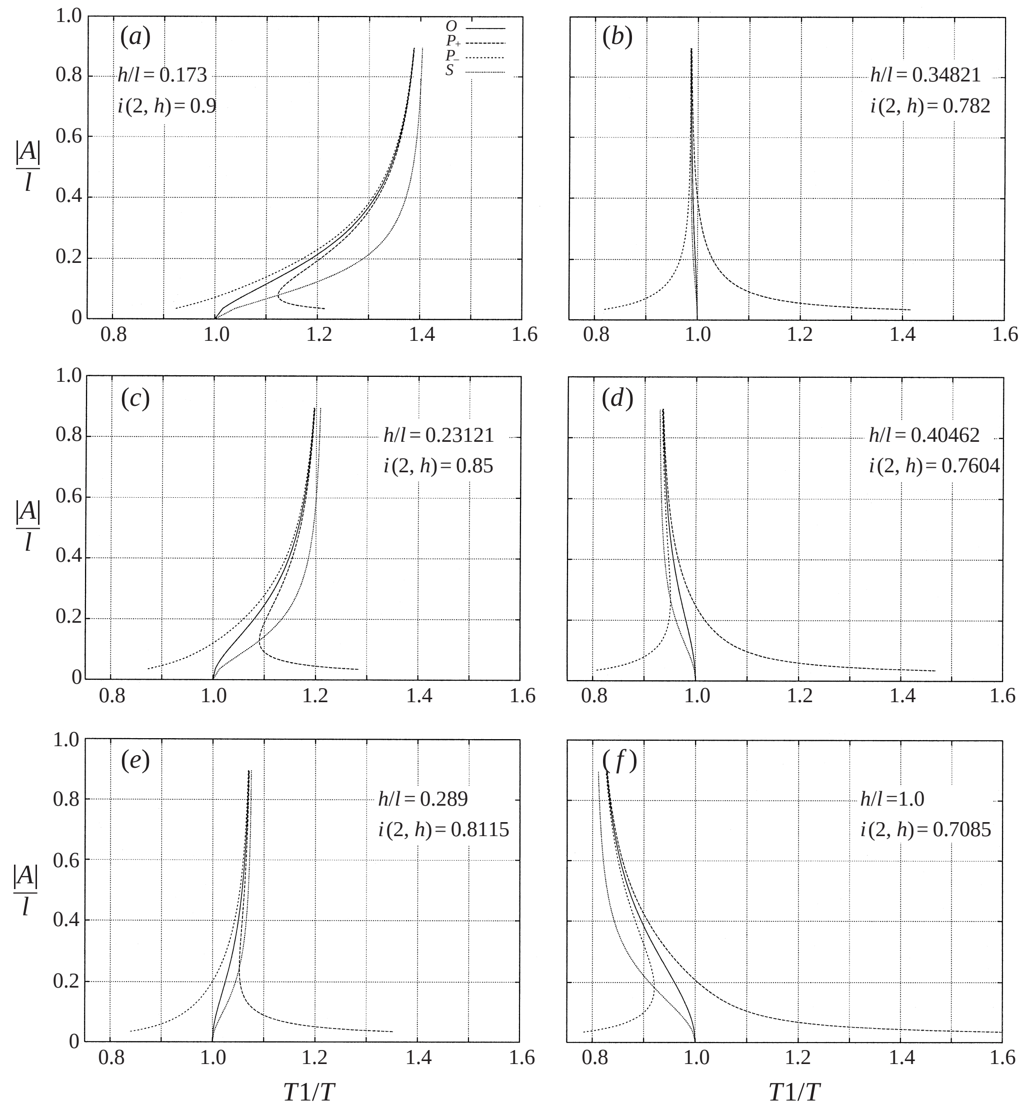

FIGURE 2. Amplitude (A)–frequency (σ) response for nonlinear sloshing due to surge excitation ($\sigma/\sigma_1 = T1/T$). h is the mean water depth, l is the tank breadth, H is the surge amplitude. $i(2,\bar{h})$ is defined by (6.9). $H/l = 0.0173$.

the tank). This can be interpreted as the amplitude–frequency dependence of free nonlinear (periodic) sloshing. The branches presented differ from diagrams obtained by Faltinsen's theory only for large values of $|A|/l$ and far away from the main resonance $\bar{\sigma}_1 = 1$. The last difference is due to the change of m_1 when varying σ. The results agree with the fifth-order theory by Waterhouse (1994) for sufficiently small amplitudes.

Similar results are obtained for steady-state sloshing in a rectangular tank excited by sinusoidal pitch motions. Let us assume the tank is pitching around the point $(0, 0, -z_0)$ of the mobile coordinate system. We can correct to $O(\epsilon)$ express that

$$\psi(t) = \psi_0 \cos(\sigma t), \quad \dot{v}_{0x} = z_0 \ddot{\psi}(t), \quad \dot{v}_{0z} = 0.$$

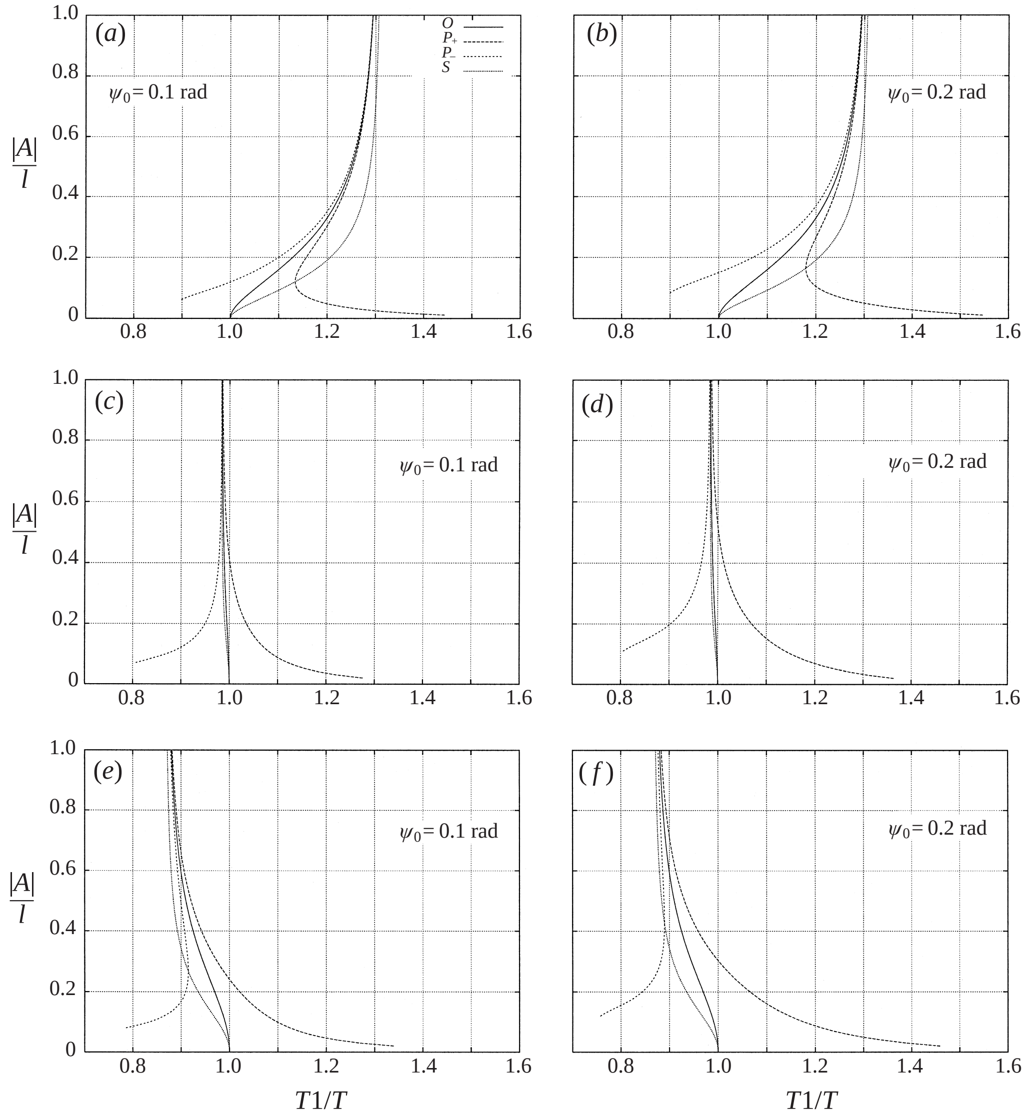

FIGURE 3. Amplitude (A)–frequency (σ) response for nonlinear sloshing due to pitch excitation ($\sigma/\sigma_1 = T1/T$). h is the mean water depth, l is the tank breadth, ψ_0 is the pitch amplitude, $(0, -z_0)$ is the position of pitch axis. $i(2, \bar{h})$ is defined by (6.9). (a, b) $z_0/l = 0$, $h/l = 0.2$, $i(2, \bar{h}) = 0.874$; (c, d) $z_0/l = 0.15$, $h/l = 0.35$, $i(2, \bar{h}) = 0.78$; (e, f) $z_0/l = 0.3$, $h/l = 0.5$, $i(2, \bar{h}) = 0.737$.

The algebraic governing equation for the frequency–amplitude response takes the following form:

$$(\bar{\sigma}_1^2 - 1)\bar{A} + m_1(\bar{\sigma}_2, \bar{h})\bar{A}^3 - \bar{P}_1\psi_0\left(\frac{z_0}{l} - \frac{S_1}{l} + \frac{g}{l\sigma^2}\right) = 0. \tag{6.8}$$

It differs from the equation of forced surge steady-state sloshing (6.5) only by the last inhomogeneous term and agrees with the corresponding equation of Faltinsen (1974).

All the results are based on the assumption that $O(\beta_1^2) = O(\beta_2)$. However, even for periodic solutions we can find a critical value of σ/σ_1 for which the amplitude of the second mode tends to infinity. It can happen for small h, or for $\bar{\sigma}_2^2 \to 4$ (see the asymptotic solution (6.3), (6.4)). In terms of σ the condition of the secondary resonance takes the form

$$\frac{\sigma}{\sigma_1} \to \sqrt{\frac{\tanh(2\pi h/l)}{2\tanh(\pi h/l)}} = i(2, \bar{h}). \tag{6.9}$$

The value $i(2, \bar{h})$ characterizes the applicability of the theory constructed (see figures 2 and 3). The ratio $T_1/T = \sigma/\sigma_1$ must be close to 1 and not close to $i(2, \bar{h})$.

Similarly, we can introduce for the third mode

$$i(3, \bar{h}) = \sqrt{\frac{\tanh(3\pi h/l)}{3\tanh(\pi h/l)}}. \tag{6.10}$$

However since $i(3, \bar{h}) < i(2, \bar{h})$, the secondary resonance is the most dangerous.

The trend of the distribution of $i(2, \bar{h})$ shows for $\bar{h}$ small enough (but large for shallow water theory) that $i(2, \bar{h}) \to 1$ as $\bar{h} \to 0$. This means that the secondary parametric resonance can occur for small depths and implies that the asymptotic theory presented is not applicable for shallow water.

The stability analysis for surge/pitch excited waves in a rectangular container was done by Faltinsen (1974). We can give reliable new treatment of the stability by introducing branches O and S in figures 2 and 3. The branch O is the relation for the frequency and amplitude for nonlinear free sloshing, which can be found from the equation

$$\text{branch } O\colon\ (\bar{\sigma}_1^2 - 1) + m_1(\bar{\sigma}_2, \bar{h})\bar{A}^2 = 0. \tag{6.11}$$

The branch O is also the asymptotic curve for P_- and P_+ as $A \to \infty$.

The branch S is the set of all turning points of the branch P_+ (or P_- for different depths) for various amplitudes $\bar{H}$ (surge excitation) or ψ_0 (pitch excitation). The turning points correspond to when (6.5) has only two solutions. The condition of two roots of equation (6.5) can be found by differentiating (6.5) with respect to A. This gives

$$\text{branch } S\colon\ (\bar{\sigma}_1^2 - 1) + 3m_1(\bar{\sigma}_2, \bar{h})\bar{A}^2 = 0. \tag{6.12}$$

The branch S does not depend on the value of the excitation amplitude and is only a function of depth–breadth ratio.

Due to the theory of bifurcations the turning point divides the branch P_+ or P_- into stable and unstable sub-branches. It was shown by Faltinsen (1974) that the upper sub-branch of P_+/P_- corresponds to unstable solutions and the lower sub-branch to stable solutions. The branch P_-/P_+ without a turning point corresponds to stable solutions. When repeating the averaging asymptotic analysis given by Faltinsen for our solutions, we arrive at the same result if $\bar{A} \ll 1$.

When varying the values of the excitation amplitude, the sub-branch situated between S and O will always correspond to unstable solutions.

7. Comparison between theory and experiments

A series of experiments on nonlinear sloshing in a smooth rectangular tank due to horizontal (surge) excitation were conducted. Figure 4 shows the tank used in the experiments. The tank has a front plate made of Plexiglas which is stiffened by two vertical L-beams. The tank was placed on a wagon that could slide back and forth

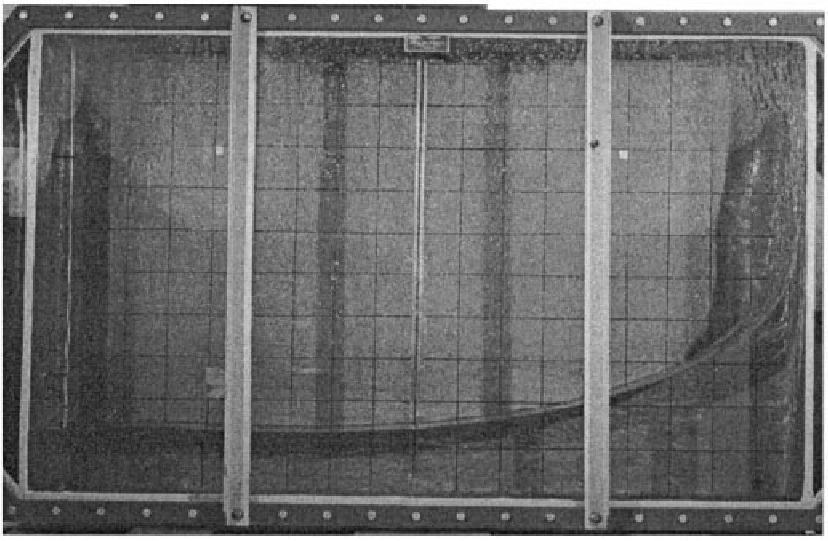

FIGURE 4. Picture of the tank.

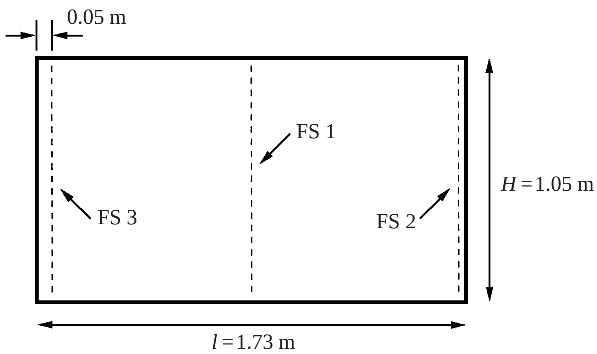

FIGURE 5. Tank dimensions and wave probe positions used in the experiments.

controlled by a hydraulic cylinder. The hydraulic system was strong enough to ensure that the motion inside the tank had little or no effect on the tank motion.

The tank height, breadth and length were respectively 1.05, 1.73 and 0.2 m. The observed free surface elevation did not vary in the length direction. The amplitude of surge excitation was between 0.02 and 0.08 m. The water depth was varied between 0.2 and 0.6 m. The tank was equipped with three wave probes, referred to as FS1, FS2 and FS3 (see figure 5). Wave probes FS1 and FS2 consist of adhesive copper tape directly placed on the tank wall. FS3 is made of steel wire and is standing 0.05 m from the left wall. The tank position was measured by a position gauge. The sampling frequency was 50 Hz and the time series were 50 s long. Video recordings and visual observation of longer simulations, up to 5 minutes, showed that steady-state oscillations with the forced oscillation period were not achieved. This implies that the dissipation in the smooth tank is very small even relative to the small damping predicted by Keulegan (1959). A reason may be that the boundary layer flow is laminar in Keulegan's experiments while it is likely to be turbulent in our case. Since transients do not die out, a beating effect occurs. The most interesting stage for analysis is during the first 50 s. During this time the beating parameters are stabilized. After this time the typical behaviour of the sloshing is repeated. Also, the preliminary analysis has shown that for beating waves of small amplitude the modulated wave is stabilized for even shorter time.

The free surface elevation had small amplitudes in the initial period after the tank was excited. In some of the tests the water was in small-amplitude motion before starting the excitation. Since the proper initial conditions are unknown two different sets are used to investigate the influence of initial conditions. One set of initial conditions is

$$\beta_i(0) = \dot{\beta}_i(0) = 0, \quad i \geqslant 1. \tag{7.1}$$

The other is based on impulse conservation. If $v_{ox} = \sigma H \cos \sigma t$, this gives

$$\beta_i(0) = 0, \quad \dot{\beta}_i(0) = -\sigma P_i H, \qquad i \geqslant 1. \tag{7.2}$$

The numerical time integrations were done by a fourth-order Runge–Kutta method and 11 equations of (5.24) were used. The simulation time on a Pentium II–366 computer was $\frac{1}{300}$ of the real time.

The examples of figures 6–9 show the effect of initial conditions on free surface elevation for different forced excitation periods T, water depth h and excitation amplitude H. So, for example in figure 6 the effect of initial conditions is not important. However, for the case of figure 7 the condition of impulse conservation leads to more reasonable description of free surface elevation. Figures 8 and 9 also demonstrate good agreement between theory and experiments. The agreement is not perfect in figure 8, but the difference between experimental and numerical simulation decreases when initial conditions are based on impulse conservation. Better agreement between theory and experiment can be achieved by realizing that the forced surge oscillation is not harmonic and does not have a constant amplitude during the initial period. This is illustrated in figure 8 where the excitation period T was not a constant during the first 12 s; it varied from 1.76 s to 1.875 s. This is caused by transient rigid body motions. We assume that these transient motions decay exponentially. This effect was simulated by varying the period and the amplitude of forced excitation in the initial phase. Figure 10 shows the effect of only varying the excitation period. A better agreement with the experiment is then achieved. Separate numerical results showed that the amplitude has less effect than variation of the frequency. The effect of varying the frequency can be found qualitatively by examining the steady-state response in figure 2.

Our theory assumes that the water does not hit the ceiling. The water touches the ceiling in the case of figure 8, but this does not have an important effect on the fluid motion. When comparing theoretical and experimental results for a case when forceful impact occurs, it is evident that they do not agree. A possible reason is energy dissipation due to the impact. The impact causes the ceiling to vibrate which represents energy loss for the fluid motion. Since the tank ceiling is very stiff in the model tests, this is unimportant in the comparative study with experiments. Furthermore, as the water hits the ceiling a jet is formed and eventually the free surface overturns and water hits the free surface. This also causes energy dissipation. An estimate of this energy loss can be calculated by using a generalization of Wagner's (1932) theory (Faltinsen & Rognebakke 1999) and assuming that the kinetic and potential energy in the jet is dissipated. An equivalent linear damping based on energy conservation can then be included in the differential equations for the generalized coordinates for the free surface. The damping coefficients $\alpha_1\dot{\beta}_1$, $\alpha_2\dot{\beta}_2$ and $\alpha_3\dot{\beta}_3$ are introduced in (5.24*a*) to

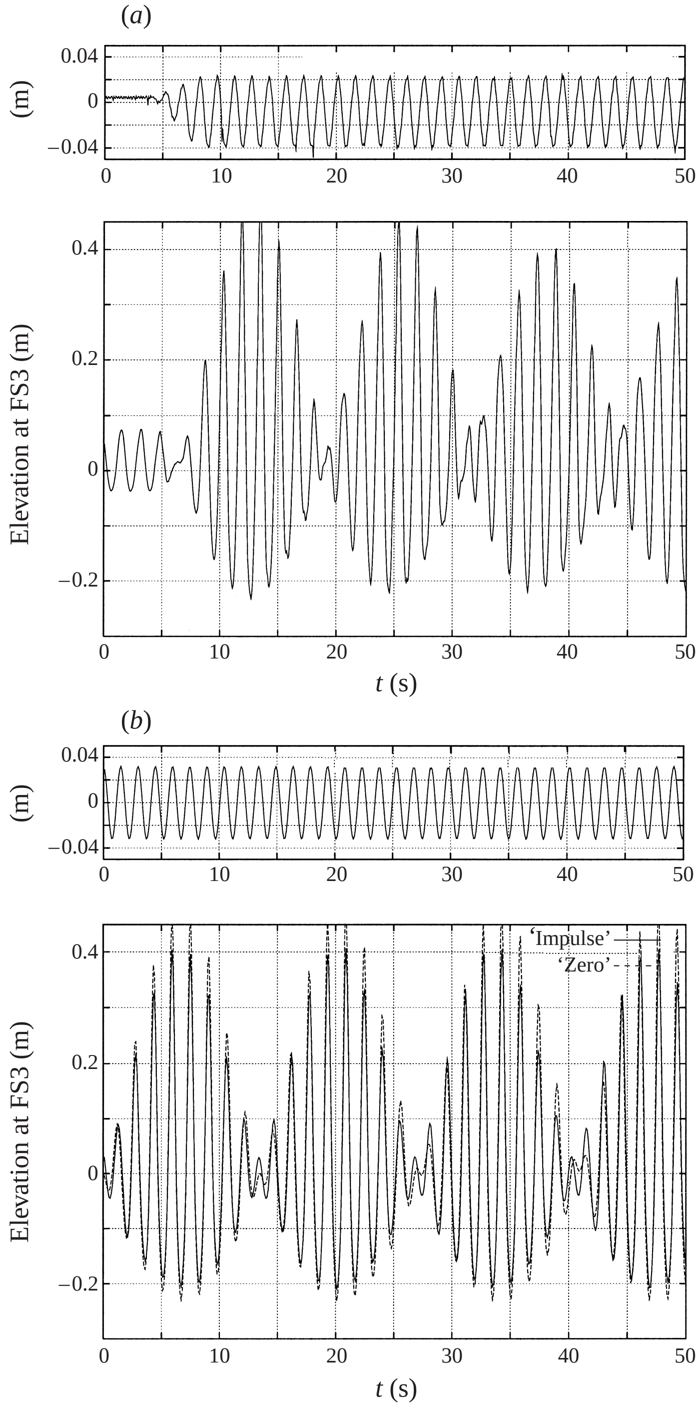

FIGURE 6. (*a*) Measured and (*b*) calculated tank position and free surface elevation at wave probe FS3 ($h = 0.6$ m, $T = 1.5$ s). The curve 'Zero' corresponds to zero initial conditions, 'Impulse' means initial impulse condition.

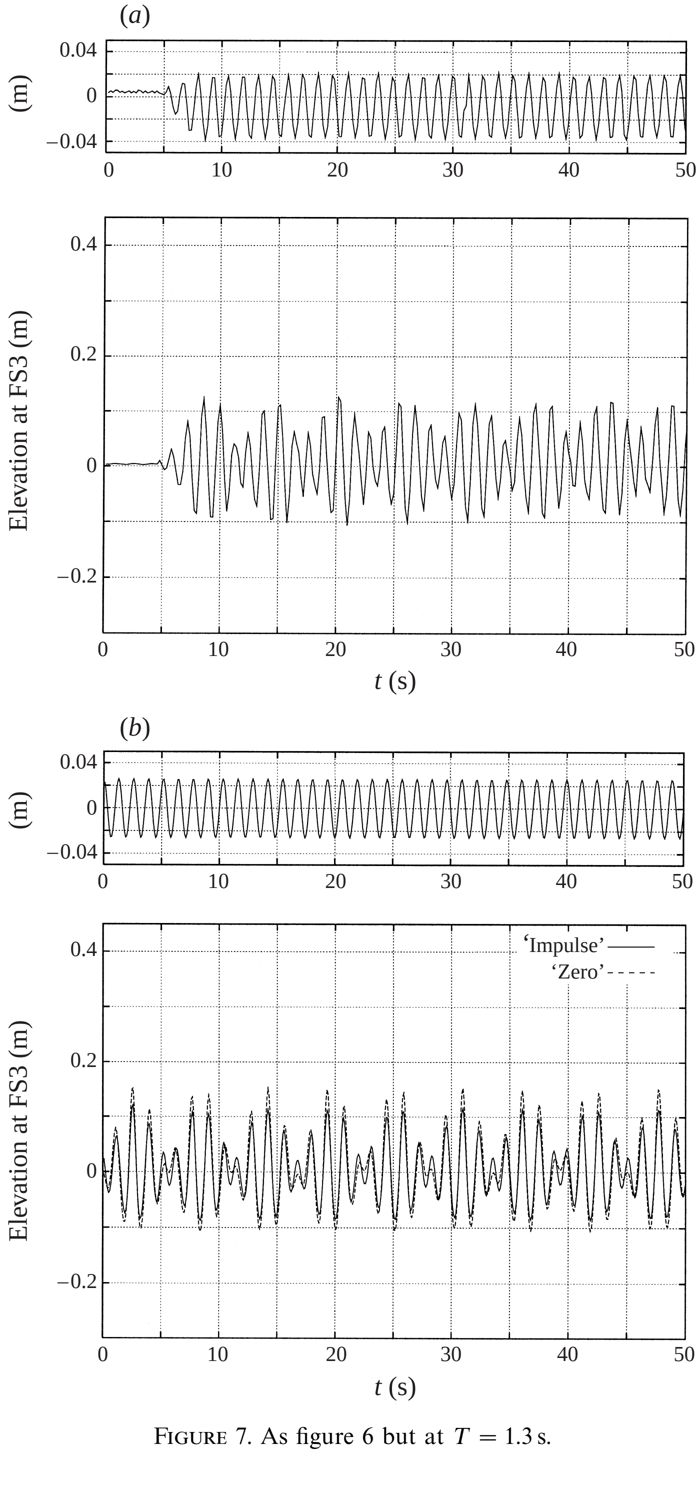

FIGURE 7. As figure 6 but at $T = 1.3$ s.

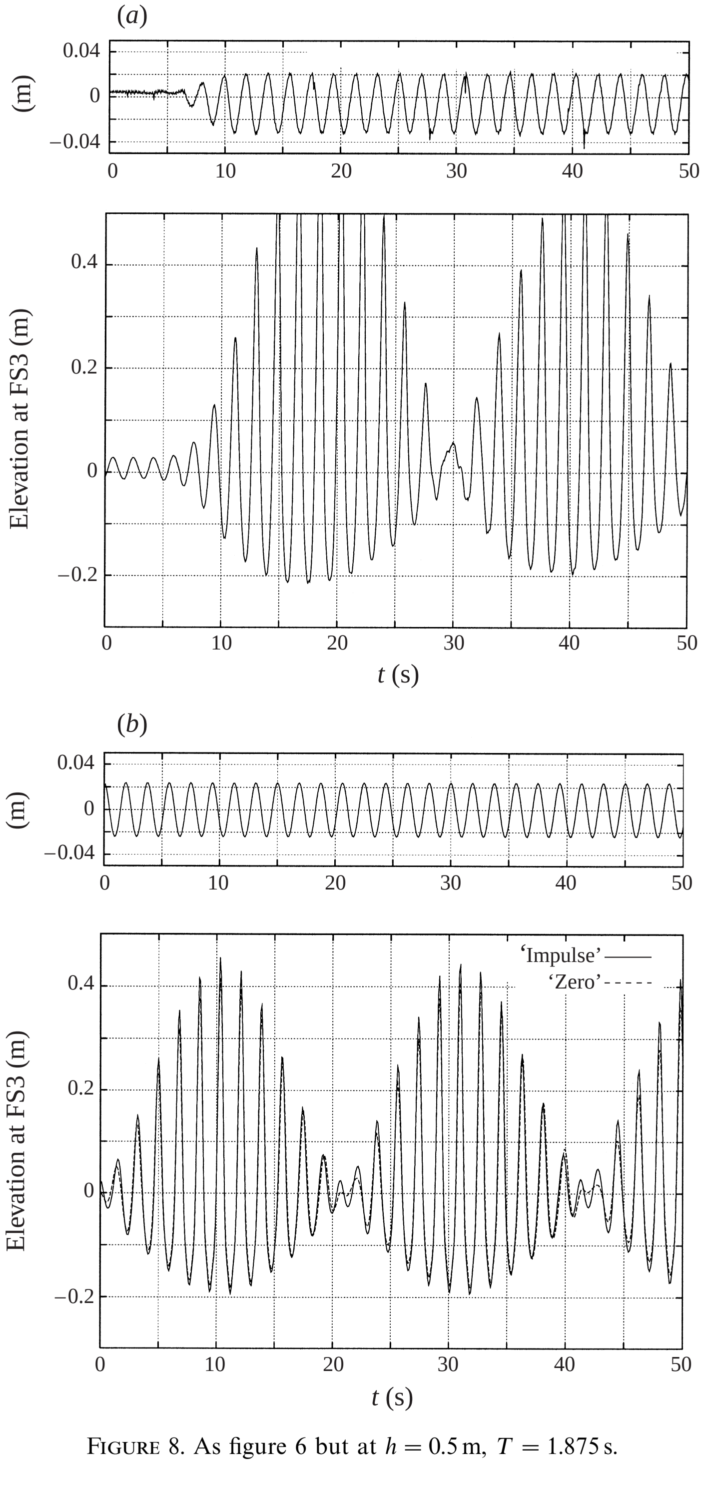

FIGURE 8. As figure 6 but at $h = 0.5\,\text{m}$, $T = 1.875\,\text{s}$.

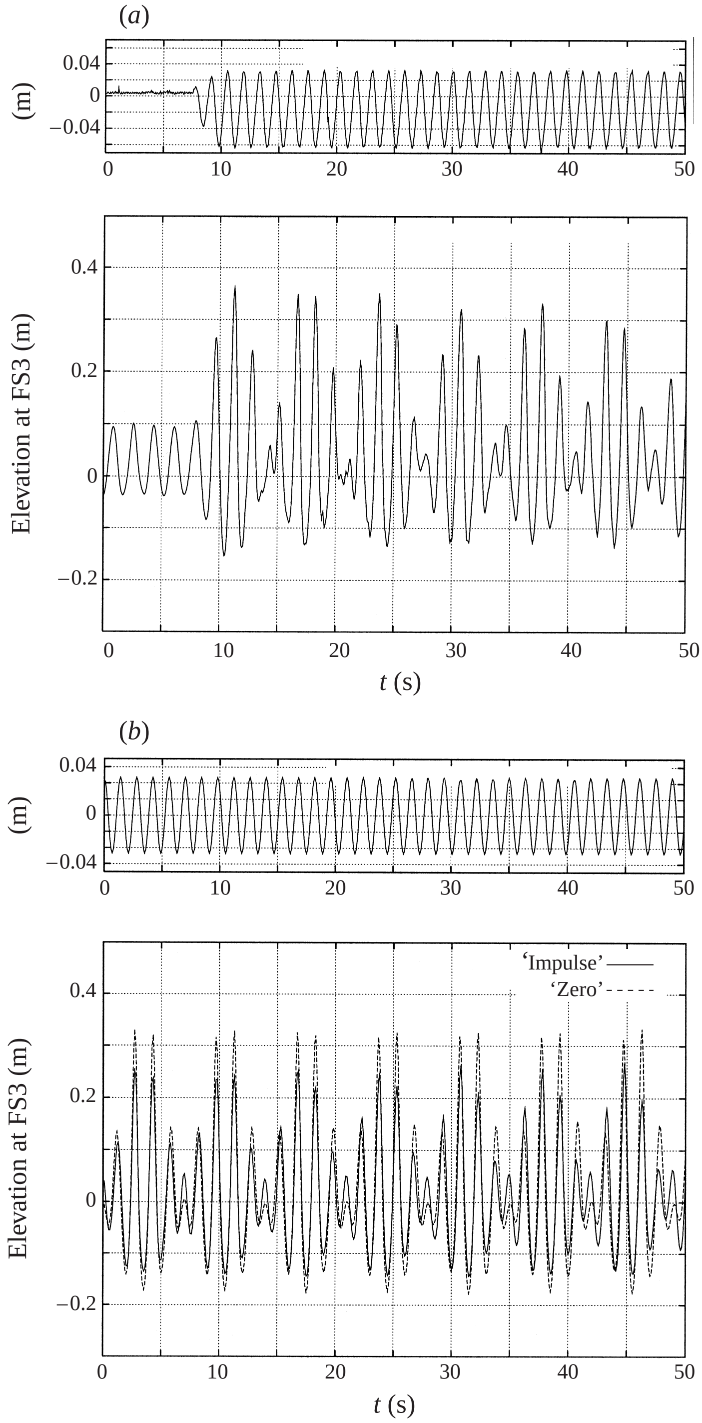

FIGURE 9. As figure 6 but at $h = 0.5$ m, $T = 1.4$ s.

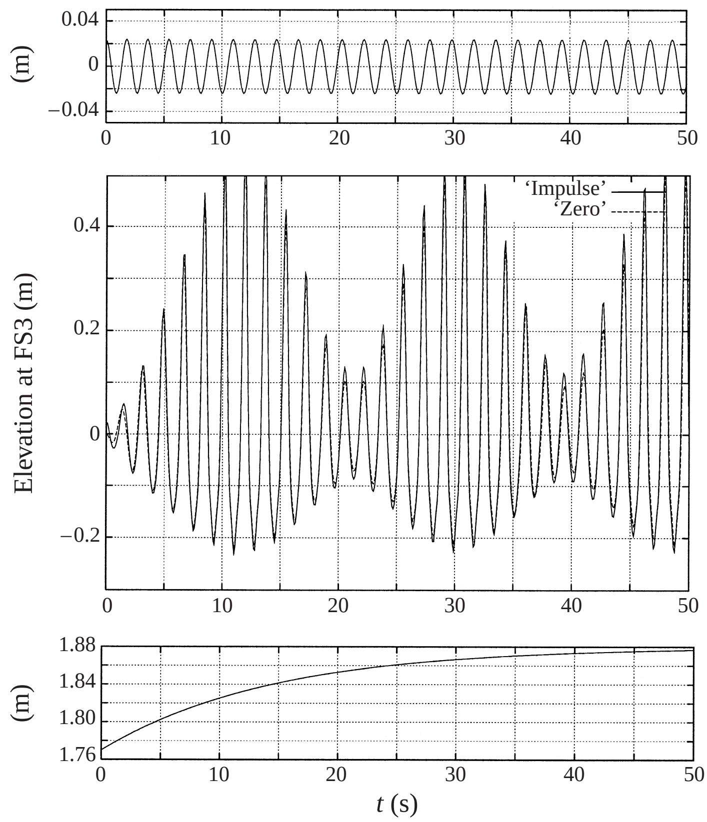

FIGURE 10. Calculated tank position and free surface elevation at wave probe FS3 for $h = 0.5$ m. Effect of varying excitation period exponentially from 1.77 s to 1.875 s.

(5.24c), respectively. Since the average forced excitation is close to the lowest natural frequency, it is only α_1 that matters. Figure 11 shows satisfactory agreement between theory and experiments by including damping. The damping will vary from cycle to cycle depending on the severity of the water impact. In the presented case we calculated approximately 40% loss of energy in the tank for every cycle due to the two impacts occurring.

The theory will break down for small water depth. Figure 12 presents experimental data and numerical simulation for $h/l = 0.173$ and $T_1/T = 0.96$. Since $i_2 = 0.9$, the effect of secondary parametric resonance is important. We note that the wave crest is well predicted, while the theoretical value for the trough is clearly lower than in the experiments. In order to improve the theoretical predictions we have to assume that at least the two lowest modes have the same order of magnitude. This means a complete change of the equation system and higher modes have to be introduced in the nonlinear equations. The introduction of the fourth mode in the nonlinear equation system will affect the difference between trough and crest so that the agreement with experiments may improve. The difference between theory and experiments is more evident in figure 13 where $T_1/T = 1.17$ and $h/l = 0.173$. The reason is once again that the primary mode is not dominating. This contradicts our theoretical assumptions. Figure 14 shows that the amplitude of the third mode is higher than the second mode, which is higher than the first mode.

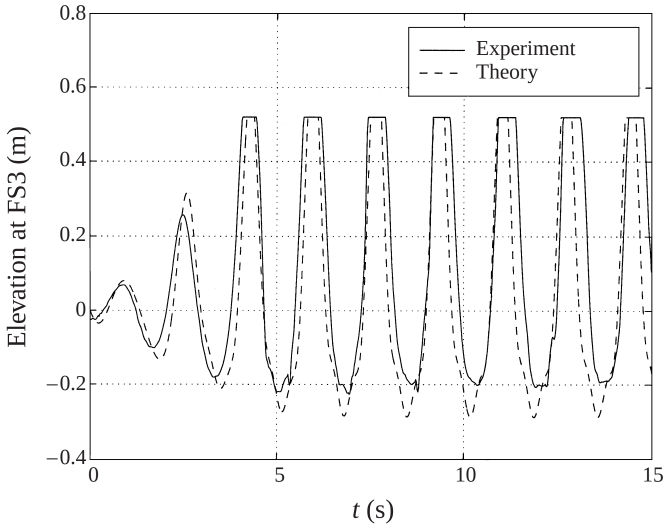

FIGURE 11. Measured and calculated free surface elevation at wave probe FS3 for $T = 1.71$ s, $h = 0.5$ m and $H = 0.05$ m. Calculations account for wave impact on tank ceiling.

8. Calculations of hydrodynamic loads on the tank

How to calculate hydrodynamic loads will be illustrated for the surge-excited rectangular tank. The general expression for the pressure is given by (2.5). By noting that $\Phi = v_{0x}x + \varphi$ and by expressing v_{0x} as $-H\sigma\sin(\sigma t)$ it follows that

$$p = p_0 - \rho\left[\frac{\partial\varphi}{\partial t} + \tfrac{1}{2}(\nabla\varphi)^2 + gz - \sigma^2 H\cos(\sigma t)x - \tfrac{1}{2}H^2\sigma^2\sin^2(\sigma t)\right]. \tag{8.1}$$

Here we use

$$\nabla\varphi = \sum_{i=1}^{N}\frac{\mathrm{i}\pi}{l}R_i(t)\left(-\sin\left(\frac{\mathrm{i}\pi}{l}\left(x+\frac{l}{2}\right)\right)\frac{\cosh((\mathrm{i}\pi/l)(z+h))}{\cosh((\mathrm{i}\pi/l)h)}, 0,\right.$$
$$\left.\cos\left(\frac{\mathrm{i}\pi}{l}\left(x+\frac{l}{2}\right)\right)\frac{\sinh((\mathrm{i}\pi/l)(z+h))}{\cosh((\mathrm{i}\pi/l)h)}\right), \tag{8.2}$$

$$\frac{\partial\varphi}{\partial t} = \sum_{i=1}^{N}\dot{R}_i(t)\cos\left(\frac{\mathrm{i}\pi}{l}\left(x+\frac{l}{2}\right)\right)\frac{\cosh((\mathrm{i}\pi/l)(z+h))}{\cosh((\mathrm{i}\pi/l)h)}, \tag{8.3}$$

where R_i and $\dot{R}_i$ are calculated by (5.12) and (5.13) and N is a number of Fourier terms ($N \geqslant 3$). When applying these formulas above the mean free surface, a Taylor expansion about the mean free surface has to be used.

The force $\boldsymbol{F}$ on the tank due to the fluid can be calculated by direct pressure integration or the compact formula derived by Lukovsky (1990)

$$\boldsymbol{F} = m_l\boldsymbol{g} - m_l[\dot{\boldsymbol{v}}_0 + \boldsymbol{\omega}\times\boldsymbol{v}_0 + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}_{1C}) + \dot{\boldsymbol{\omega}}\times\boldsymbol{r}_{1C} + 2\boldsymbol{\omega}\times\dot{\boldsymbol{r}}_{1C} + \ddot{\boldsymbol{r}}_{1C}] \tag{8.4}$$

where $\boldsymbol{r}_{1C}$ is radius-vector of the mass centre in mobile coordinate system $Oxyz$ and m_l is fluid mass. We note that $\boldsymbol{F}$ includes the static force component $m_l\boldsymbol{g}$ in addition to hydrodynamic forces.

The formula takes the form

$$\boldsymbol{F} = m_l\boldsymbol{g} - m_l(\dot{\boldsymbol{v}}_0 + \ddot{\boldsymbol{r}}_{1C}) \tag{8.5}$$

in the absence of angular motions ($\omega = 0$).

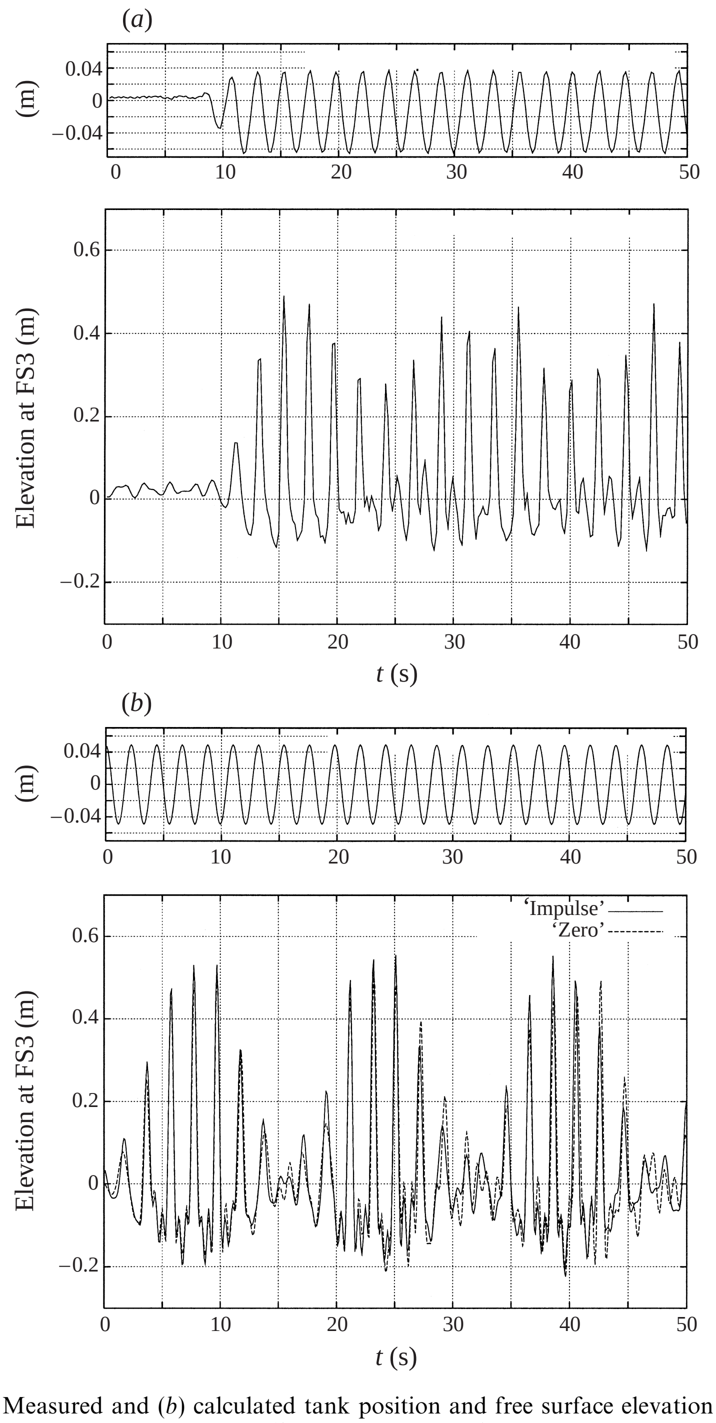

FIGURE 12. (*a*) Measured and (*b*) calculated tank position and free surface elevation at wave probe FS3 ($h = 0.3$ m, $T = 2.2$ s).

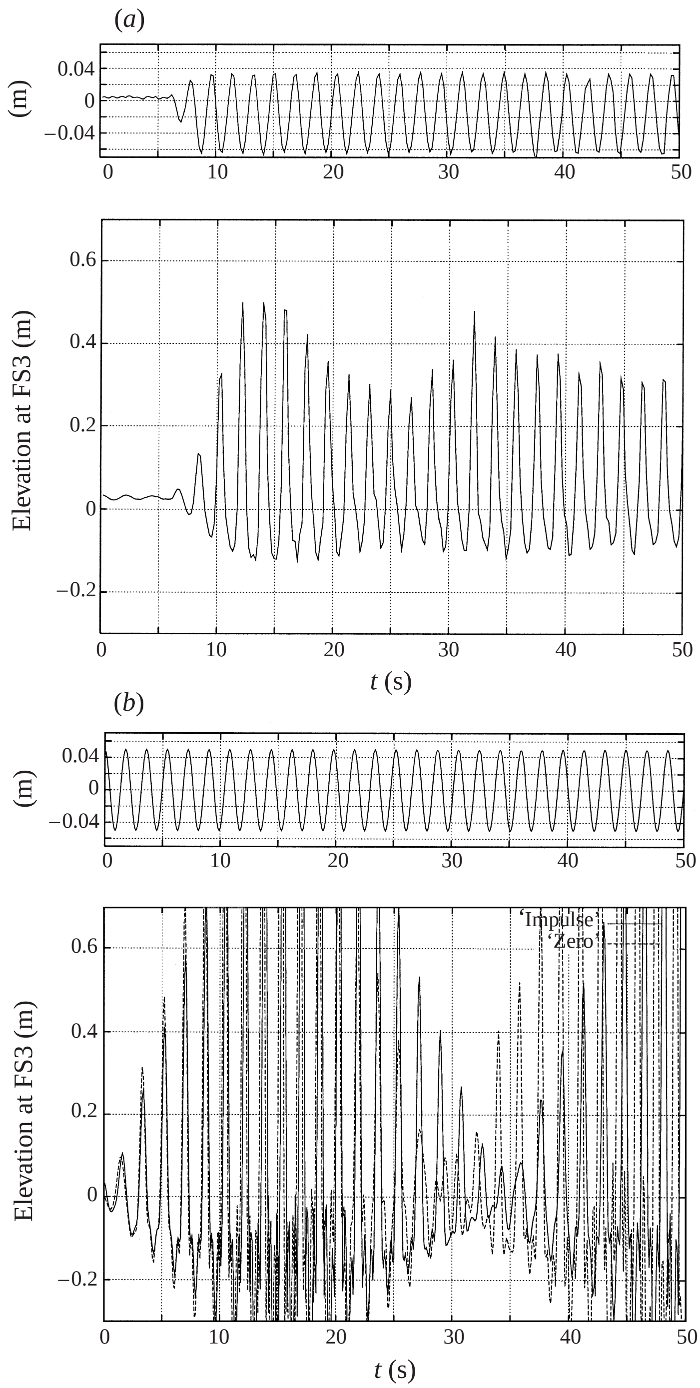

FIGURE 13. As figure 12 but at $T = 1.8$ s.

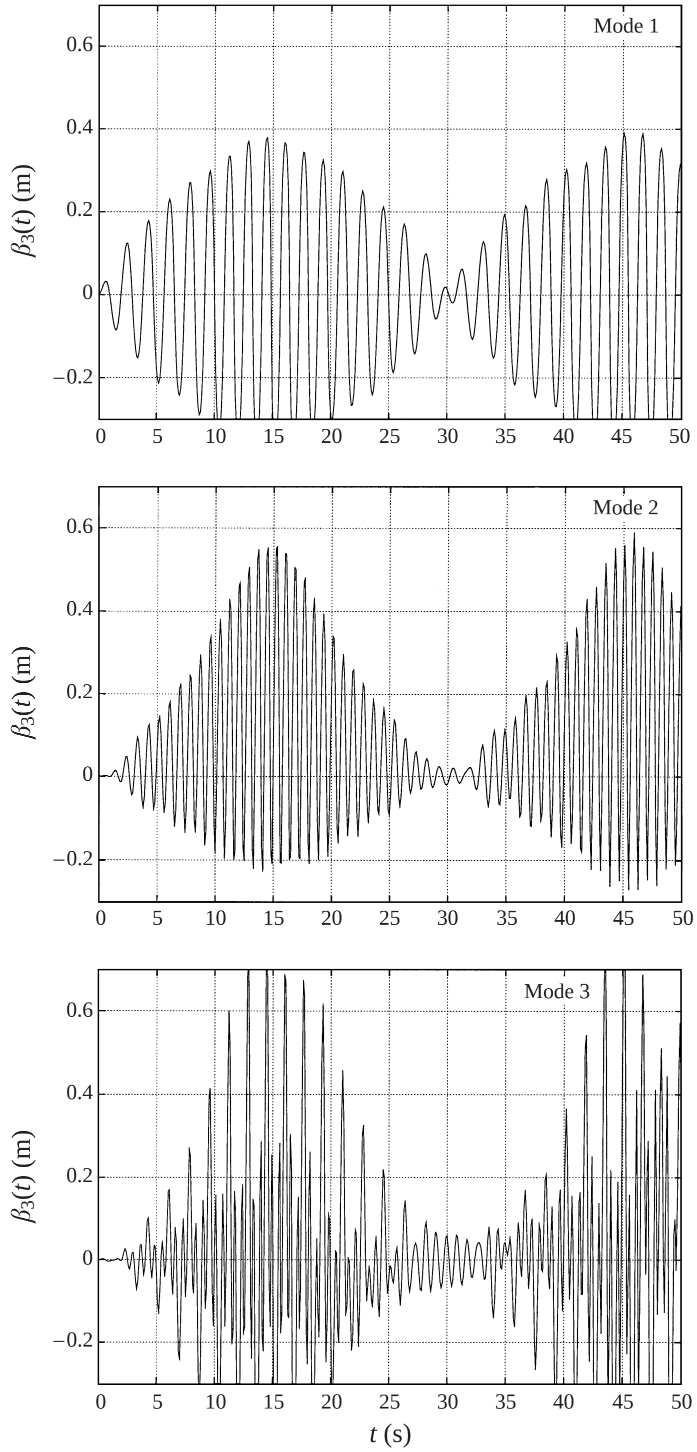

FIGURE 14. Contribution of the three lowest modes to the calculated free surface elevation at wave probe FS3; $h = 0.3$ m, $T = 1.8$ s.

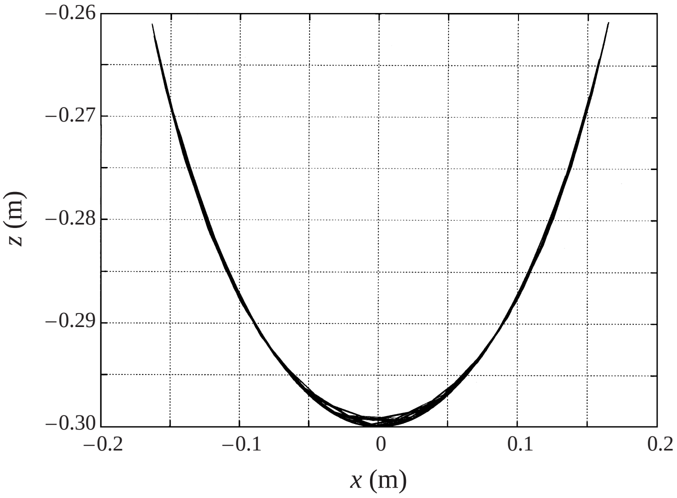

FIGURE 15. The position of mass centre for the case in figure 6.

The calculation shows, that if $\boldsymbol{r}_{1C} = (x_C(t), 0, z_C(t))$, then

$$x_C = -\frac{l}{\pi^2 h}\sum_{i=1}^{N}\beta_i(t)\frac{1}{i^2}(1+(-1)^{i+1}), \qquad z_C = -\frac{h}{2}+\frac{1}{4h}\sum_{i=1}^{n}\beta_i^2(t), \tag{8.6}$$

where the point $(0, -h/2)$ corresponds to mass centre of unperturbed fluid.

By introducing the vector $\boldsymbol{F} = (F_x, 0, F_z)$ we arrive at

$$\left.\begin{aligned} F_x/m_l &= \left(H\sigma^2\cos\sigma t + \frac{l}{\pi^2 h}\sum_{i=1}^{N}\ddot{\beta}_i(t)\frac{1}{i^2}(1+(-1)^{i+1})\right), \\ F_z/m_l &= -\left(g+\frac{1}{2h}\sum_{i=1}^{N}(\ddot{\beta}_i\beta_i+\dot{\beta}_i^2)\right). \end{aligned}\right\} \tag{8.7}$$

Figure 15 shows the trajectory of the mass centre. Figure 16 presents the trajectory of the end of the vector $\boldsymbol{F}/m_l$.

The hydrodynamic moment $\boldsymbol{N}$ on the tank can also be calculated by the special formula derived by Lukovsky (1990) (moment axis coincides with Oy)

$$\begin{aligned} \boldsymbol{N} = m_l\boldsymbol{r}_{1C}\times(\boldsymbol{g}-\boldsymbol{\omega}\times\boldsymbol{v}_0-\dot{\boldsymbol{v}}_0)-\boldsymbol{J}^1\cdot\dot{\boldsymbol{\omega}}-\dot{\boldsymbol{J}}^1\cdot\boldsymbol{\omega}-\boldsymbol{\omega}\times(\boldsymbol{J}^1\cdot\boldsymbol{\omega}) \\ -\ddot{\boldsymbol{l}}_\omega+\dot{\boldsymbol{l}}_{\omega t}-\boldsymbol{\omega}\times(\dot{\boldsymbol{l}}_\omega-\boldsymbol{l}_{\omega t}), \end{aligned} \tag{8.8}$$

where the inertia tensor $\boldsymbol{J}^1$ is defined by (3.13) and $\boldsymbol{l}_\omega, \boldsymbol{l}_{\omega t}$ by (3.15).

For $\boldsymbol{\omega} = \boldsymbol{0}$

$$\boldsymbol{N} = m_l\,\boldsymbol{r}_{1C}\times(\boldsymbol{g}-\dot{\boldsymbol{v}}_0)-\ddot{\boldsymbol{l}}_\omega+\dot{\boldsymbol{l}}_{\omega t}. \tag{8.9}$$

The time-varying functions $\boldsymbol{l}_\omega, \boldsymbol{l}_{\omega t}$ depend on the solutions of the Neumann boundary value problem (3.5) and can be expressed mathematically like the Stokes–Zhukovsky potentials.

By using Green's formula we get

$$N(t) = m_l(x_C g - z_C\dot{v}_0) - \rho\frac{\mathrm{d}}{\mathrm{d}t}\int_{S+\Sigma}(zv_1 - xv_3)\varphi\,\mathrm{d}S, \tag{8.10}$$

where $\boldsymbol{N} = (0, N(t), 0)$.

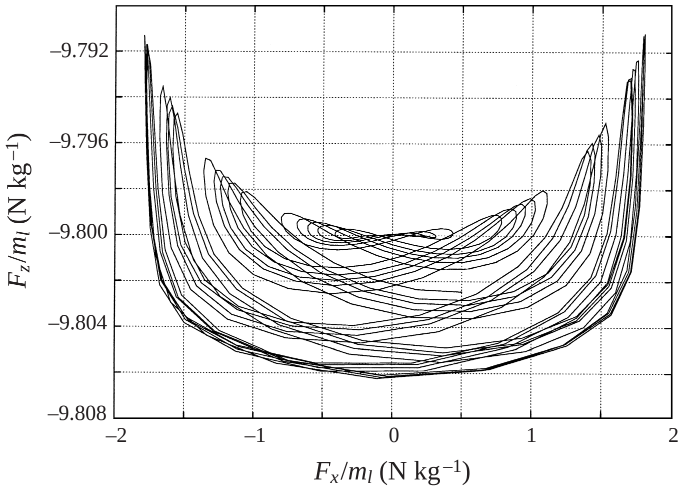

FIGURE 16. The trajectory of the vector of the calculated hydrodynamic force $(F_x/m_l, F_z/m_l)$ on the tank for the case in figure 6.

This is not as simple as the formula (8.7) for the force, but is useful in a verification procedure by comparing with the direct pressure integration of the moment. This should in both cases be derived correct to $O(\epsilon)$.

9. Conclusions

I. Using the Bateman–Luke variational principle, we generalize the procedure proposed by Miles (1976) and Lukovsky (1976) to derive a modal system describing nonlinear sloshing of an incompressible perfect fluid with irrotational flow partly occupying a tank performing an arbitrary three-dimensional motion. If the tank has vertical walls near the mean free surface, this procedure leads to an infinite-dimensional system of nonlinear differential equations coupling the generalized time-dependent coordinates. No assumptions about the order of smallness are made. It applies to any type of rigid body motion. The surface and domain modes do not need to be natural modes. This means that the multidimensional modal discrete system derived has the most general form of the modal equations and can be used for modelling different 'fluid–structure' problems including the problems associated with transient sloshing and coupled 'ship–fluid cargo' motions.

II. Two-dimensional sloshing in a rectangular smooth tank with finite water depth has been studied theoretically. The tank is oscillating with arbitrary rigid body motions of small magnitude with an average frequency close to the lowest natural frequency of the fluid motion. A finite-dimensional asymptotic model with multiple degrees of freedom is derived. This is based on the general discrete infinite-dimensional modal model. The lowest mode is assumed dominant. Each mode has different order of magnitude. The three lowest modes are interacting nonlinearly with each other. An important feature relative to other established nonlinear theories is that transient effects can be described. Since the theory is expressed in terms of a set of nonlinear ordinary differential equations in time, it is considerably simpler than a direct numerical solution of the fluid motion.

Periodic solutions are studied analytically. The amplitude–frequency response is consistent with the fifth-order steady-state solution by Waterhouse (1994).

It is shown that the theory is not valid when the water depth (h) becomes small relative to the tank breadth (l). This is due to secondary parametric resonance. It is then necessary to include nonlinearly interacting modes having the same order of magnitude. This is demonstrated for a tank with $h/l = 0.173$.

III. We have conducted experimental studies of the free surface elevation for forced surge oscillations of two-dimensional flow in a rectangular tank. It is demonstrated experimentally that it takes a very long time for transient fluid motion to die out. This did not occur during an observation period of 5 minutes, which corresponds to the order of 150–200 oscillations in terms of the excitation period. The consequence is that steady-state solutions of nonlinear sloshing in a smooth tank can have limited applicability. Modulated ('beating') waves occurred as a consequence of transient and forced oscillations. The amplitude/'beating' period was stabilized during the first 50 s.

Since we could not exactly state what the initial conditions were in the experiments, a sensitivity study was performed with different initial conditions in the theoretical model. The results were not strongly dependent on this, but better agreement between theory and experiments was in general obtained by using an initial condition based on impulse conservation. For several experiments we observed fluctuations of the excitation frequency in an initial period up to approximately 10 s. This effect was important to include in the theoretical model. There is good agreement with experimental free surface elevation when $h/l \geqslant 0.28$.

IV. The theory was compared with experiments when forceful water impact on the tank ceiling occurred. The theory assumes no tank ceiling. The experimental free surface elevations showed a clear influence of the water impact. It was speculated that this was due to energy dissipation and phenomenological linear damping terms were introduced in the discrete modal system. Good agreement with the experiments was demonstrated. This is an area of future research.

V. It is shown how hydrodynamic forces on the tank can be calculated in a simple way. An alternative formula for the hydrodynamic moment is also presented. The form of the expressions facilities simulations of a coupled 'vehicle–fluid' system.

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